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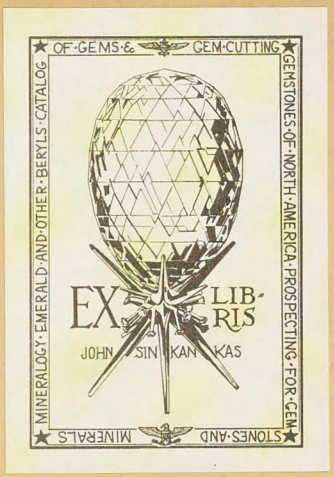

THE STEREOGRAPHIC PROJECTION AND ITS POSSIBILITIES, FROM A GRAPHICAL STANDPOINT,

WITH 34 FIGURES AND 4 PLATES;

BY

SAMUEL L. PENFIELD,

Professor of Mineralogy in the Sheffield Scientific School of Yale University.

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PREFACE.

This pamphlet is a reprint of an article which first appeared in the January and February numbers of the American Journal of Science and Arts, 1901. The success which the writer has had in applying graphical methods to the solution of problems in crystallography has led him to believe that others will find the methods equally useful, not only in crystallography, but also in geography, navigation, astronomy and wherever spherical trigonometry may be employed. An advantage of the methods is that they enable those who have never studied spherical trigonometry to solve problems intelligibly, and wholly without the use of formulas and tables. Moreover, in most instances the results obtained by plotting will be as exact as the reliability of the available data demand, while by plotting on a sufficiently large scale almost any desired degree of accuracy may be secured.

New Haven, February 1, 1901.

THE

STEREOGRAPHIC PROJECTION.

FROM A GRAPHICAL STANDPOINT.

Introduction.—The results which are given in the present paper are the outgrowth of a desire on the part of the writer to simplify some of the processes of plotting and determining crystal forms. The whole subject of stereographic projection, as it has gradually unfolded itself to him during the past two years, has revealed so many possibilities, and seems so important and of such general interest, that it has been decided to present first a paper treating of the stereographic projection alone, leaving for a later communication its applications to special problems of crystallography.

As far as the mathematical principles of the projection are concerned, the writer lays claim to no new facts. The projection is treated, in more or less detail (usually very briefly), in most text-books of crystallography, and instructions are given for making stereographic projections. The processes recommended, however, are generally tedious, and one of the objects of the present paper is to indicate how projections may be constructed easily and very accurately. Moreover, no mathematical formulas nor equations have been used in developing the subject, neither have tables been employed other than one of natural tangents for calculating a certain scale. The principles of the projection, as set forth in this article, are absolutely exact; while the errors involved in solving problems by graphical methods are dependent upon one's ability to locate points and read scales correctly, the errors generally diminishing as the size of the projection increases. It is also true of numerical calculations that the processes are limited. Given exact data, results accurate to the minute or to the second are

obtained according as four-place or seven-place logarithm tables are employed ; while for some very exact geodetic computations, where small fractions of a second must be taken into consideration, ten-place logarithm tables are at times made use of. The advantages of graphical methods over numerical calculations are numerous, and are fully appreciated by engineers and others who deal extensively with measurements and practical results derived therefrom.

The writer would be one of the last to claim that numerical calculations can be dispensed with, yet he contends that, for a large number of problems, especially those where the data are not very exact, results obtained by graphical methods are in every way as serviceable as those secured by calculation. Then, too, it is possible to make computations by graphical methods wholly without the use of formulas and tables, and the processes can be carried out intelligently by persons who have had no special mathematical training, provided only that they have an appreciation of measurements expressed in terms of degrees and fractions. Many advantages to be derived from the use of the stereographic projection will naturally suggest themselves during the course of this paper. In subsequent paragraphs some of these advantages will be set forth, and results obtained by plotting will be given, in order that an idea of the accuracy of the method may be obtained.

General Principles of the Stereographic Projection.—In this method of projection all points and arcs on the surface of a sphere are projected on a flat surface (the plane of the projection) passing through the center of the sphere, the pole or point to which everything is projected being located on the surface of the sphere and at right angles to the plane of the projection. Often the equatorial plane is chosen as the plane of the projection, and the pole to which everything is then projected is the south pole. Under the foregoing conditions, it is also customary to represent only the features of the upper half of the sphere (the northern hemisphere) in the projection, although, as will be shown, the projection may be carried out beyond the equator so as to include the southern hemisphere as well. Projections are likewise frequently made upon a plane passing through some north and south meridian, in which case the pole of the projection will be located upon the equator, at right angles to the plane of the projection. As will be shown, projections can be made without difficulty upon any desired plane. A most important feature of the stereographic projection is that all angular distances and directions, which can be plotted and measured only with difficulty on a spherical surface, appear on the flat surface of the stereographic projection in such relations that they may be easily plotted and measured. This is true of no other method of projection.

Some essential features of the stereographic projection are illustrated in figure 1. The circle represents a vertical section

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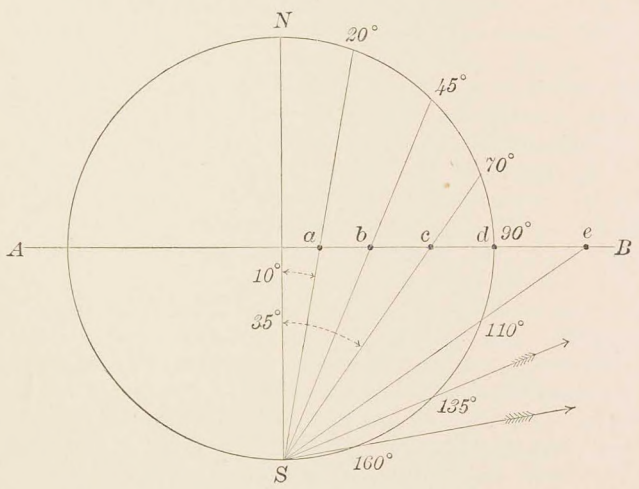

through a sphere, or a north and south meridian if considered as such. *N* and *S* are the north and south poles, respectively, and *A B* is the trace of the plane of the equator. Points on

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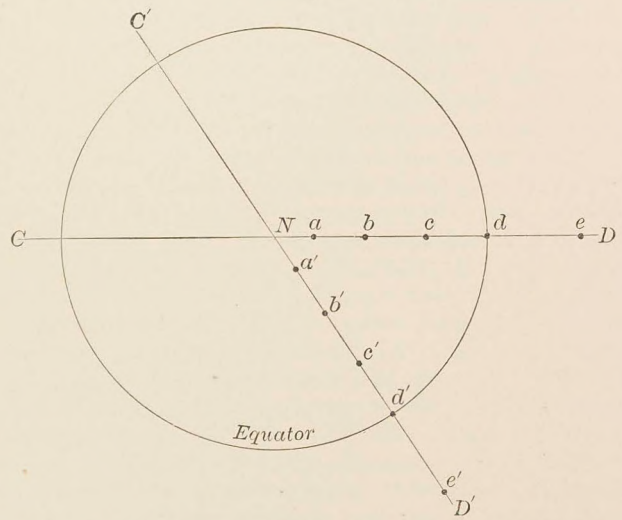


the meridian 20°, 45°, 70°, and 90° from *N*, when projected to the south pole, are seen upon the plane of the equator at *a*, *b*,

c, and *d*, respectively, while points more than 90° from *N* (110°, 135°, and 160°, for example), when projected to the south pole and continued along the lines of projection until they meet the plane of the equator, appear beyond the circle at *e*, and points still farther out as indicated by the direction of the arrows. In stereographic projection, figure 2, the equator appears as a circle, the north pole *N* occupies a position in the center, and a north and south meridian is projected upon the plane of the equator as a straight line corresponding to some diameter of the circle, it may be *C D*, or it may have some other direction, *C′ D′* for example, depending, so to speak, upon the longitude of the meridian. Points 20°, 45°, 70°, 90°, and 110° from *N*, as measured on a north and south meridian, appear in stereographic projection, figure 2, at *a*, *b*, *c*, *d*, and *e*, or *a′*, *b′*, *c′*, *d′*, and *e′*, respectively, the distances of these points from *N* being equal to those of corresponding points from the center, figure 1, provided that the diameters of the two circles are the same. The diameter *C D*, figure 2, represents a stereographically projected north and south meridian, and the distances *N* to *a*, *b*, *c*, *d*, and *e*, indicate 20°, 45°, 70°, 90°, and 110°, respectively, as measured on the meridian. The true linear distances *N* to *a*, *b*, etc., are equal to the tangents of half the angles under consideration, the radius of the circle being regarded as unity. The foregoing tangent relation is well illustrated in figure 1, and depends upon an important principle of geometry; namely, that two lines within a circle meeting at the circumference (as any two lines meeting at *S*, figure 1) make an angle with one another which is measured by half the arc included between the lines at the circumference. The distances *N* to *a*, *c* to *d*, and *d* to *e*, each represent 20° in stereographic projection, and the relations of these distances should be carefully considered. Proceeding from the center, each stereographically projected degree is somewhat greater than the one just before it, hence the distance *c* to *d* just within the equator is nearly twice as great as *N* to *a* near the pole, and *d* to *e* just beyond the equator is more than twice as great as *N* to *a*.

From a consideration of figure 2, it is evident that, although any point on the stereographic projection (*a*, for example) which is 20° from the pole has a fixed relative distance from the center irrespective of the size of the circle, the absolute distance of 20° from the pole will vary with the size of the circle. Hence, the construction of stereographic projections can be greatly facilitated by adopting some definite size for the fundamental circle, and devising certain protractors and scales, by means of which points occupying known positions on the sphere may be quickly and accurately plotted. In

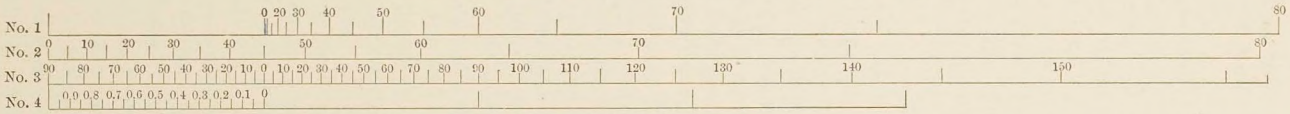
deciding upon an appropriate scale on which to make stereographic projections, it was necessary carefully to consider the two following points:

If the construction is too small, it is difficult to locate given points with sufficient accuracy, by means of a pencil and ordinary drawing instruments; while, on the other hand, if the construction is on too large a scale, the drawing becomes unwieldy and presents many difficulties, although the degree of accuracy is greatly increased. After some experimenting it was decided to make the projections on a circle of 14^{cm} diameter, and after nearly two years' experience, during which time almost all kinds of crystallographic problems have been under consideration, it may be stated that this scale has proved very satisfactory.

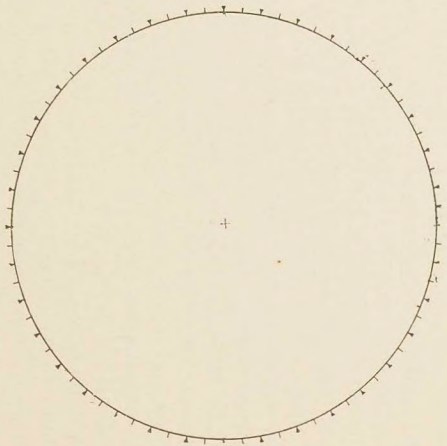
The Graduated Circle.—As one of the first aids for the quick and accurate construction of stereographic projections, a circle of 14^{cm} diameter, graduated into degrees, was engraved. Every tenth degree of the graduation is accentuated, so that it may be quickly caught by the eye, but it was thought best not to number the entire graduation, as degrees can be easily counted off. The exact center of the circle is indicated by a small cross. The circle was engraved by means of a dividing engine, which is equivalent to a guarantee of its accuracy. (See page 23.) The circle and some scales which will be described later are shown, much reduced and with only 5° graduation, in figure 3. They are printed together on large sheets of paper of excellent quality, the idea being that the sheets may be purchased at a trifling expense, and used for plotting all kinds of problems in stereographic projection.

Protractor No. I for plotting Stereographic Projections.—In order to facilitate the work of locating known points on a diameter of the graduated circle, a special protractor, designated as No. I, has been devised, which is shown without reduction in figure 4. The semicircle, divided into degrees, has a diameter of 14^{cm}, thus corresponding to the diameter of the graduated circle, figure 3, of the printed sheets. Holding the protractor in a vertical position, and regarding the upper 0°–90° point of the semicircle as the north pole, lines drawn from degree points on the semicircle to an imaginary south pole would cross the diameter at correspondingly numbered, *stereographically projected*, degree points. The distance from the center to each degree line of the graduation was determined by calculation, and the scale was then engraved by means of a dividing engine. The graduation has been numbered in both directions in order that angles given from either the pole or the equator can be conveniently located. The protractor is printed on cardboard and is inexpensive. It may be

DIVIDED CIRCLE AND SCALES FOR PLOTTING STEREOGRAPHIC PROJECTIONS.

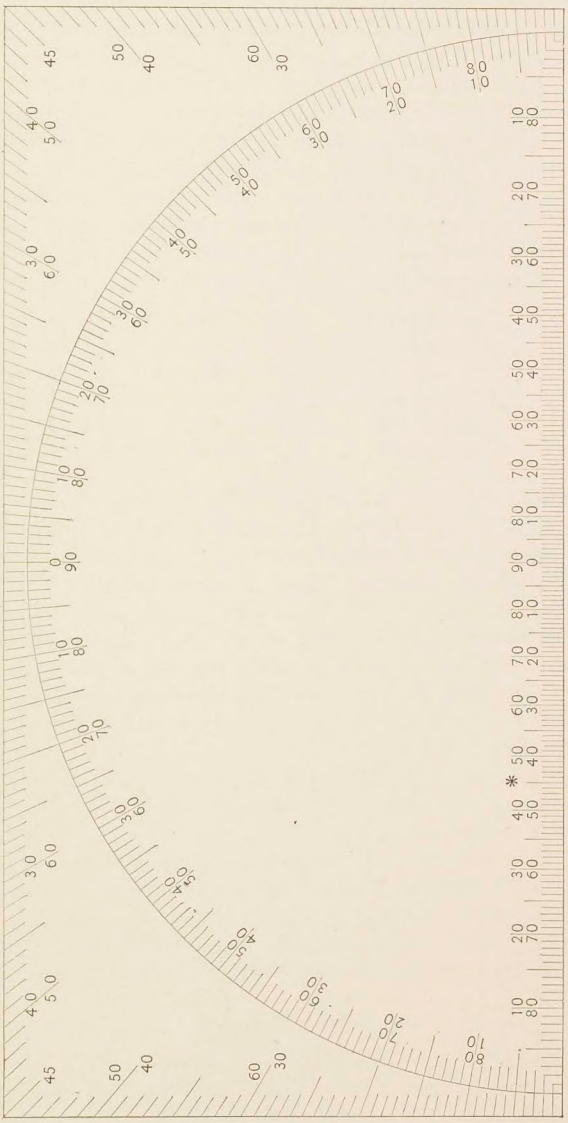


No. 1 Radii of Stereographically projected arcs of great circles, degrees measured from the divided circle.
No. 2 Radii of Stereographically projected arcs of small vertical circles, degrees measured from the divided circle.
No. 3 Degrees of a vertical great circle stereographically projected on a diameter.
No. 4 Decimal parts of the radius of the divided circle.



On the original engine-divided plate the circle has a diameter of 14^{cm} and is divided into degrees (compare plate I). The scales likewise are subdivided so as to give desired parts to degrees.

employed either in semicircular or rectangular form as an ordinary protractor for laying off and measuring plane angles.



PROTRACTOR No I, FOR PLOTTING STEREOGRAPHIC PROJECTIONS.
The graduation on the base line gives the stereographically projected degrees.
From * to 0 equals the chord of 90°.
Printed from the original engine-divided plate.

Scale No. 3, accompanying the graduated circle, figure 3, repeats the graduation of the base line of the protractor, and the graduation is further continued for arcs of more than 90°

from the pole. Thus, referring to figures 1 and 2, p. 3, the distances at which the lines of projection of arcs of 110° and 135° would meet the diameter can be taken directly from this scale. Various examples of the uses of scale No. 3 will appear during the course of this article.

Owing to the size of the sheets upon which the divided circle and scales are printed, there is a limit to the number of stereographically projected degrees which can be given. As seen in figure 3, the end of scale No. 3 indicates the 156th degree from the pole. Since in some operations it may be necessary to carry the projection beyond the limits of this scale, the distances from the pole to the remaining projected degrees are given in millimeters in the following table:

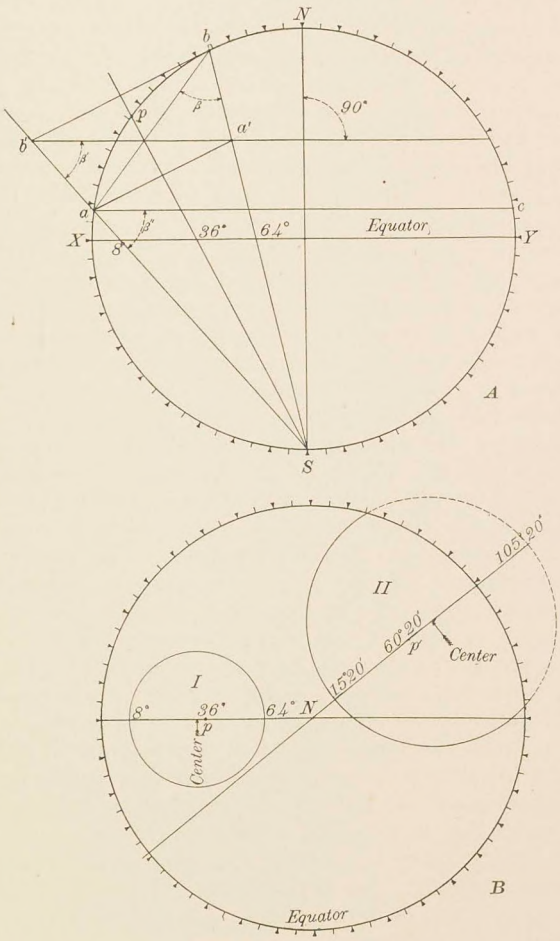
157°–344·1mm	163°–468·4mm	169°– 727·0mm	175°–1603·3mm
158 –360·1	164 –498·1	170 – 800·1	176 –2004·5
159 –377·7	165 –531·7	171 – 889·4	177 –2673·2
160 –397·0	166 –570·1	172 –1001·0	178 –4010·3
161 –418·3	167 –614·4	173 –1144·5	179 –8021·2
162 –442·0	168 –666·0	174 –1335·7	180 Infinity

The Possible Circles on a Spherical Surface.—About a point p located anywhere on the surface of a sphere, two kinds of circles may be described; an indefinite number of *small circles*, whose distance from p is less than 90°, and one great circle, at a distance of 90° from p. If p is located at either the north or the south pole of a sphere, the small circles described about p correspond to the parallels of latitude of a terrestrial globe, and the great circle answers to the equator. If the sphere is oriented with its north and south poles in a vertical direction, and p is located on the equator, the small circles described about p will have a vertical position and will be referred to as *vertical small circles.* A *vertical great circle*, on the other hand, will pass through the north and south poles, and will thus correspond to some north and south meridian of a terrestrial globe. A great circle, whatever its position, corresponds to some circumference of a sphere, and, moreover, every great circle has this peculiarity, that it crosses the horizontal great circle, or equator, at two points which are antipodal.

The Stereographic Projection of Small Circles.—The upper portion A of figure 5 is intended to represent a vertical section through the center of a sphere; hence, the graduated circle corresponds to some north and south meridian. $X\ Y$ is the trace of the plane of the equator, and N and S are the north and south poles, respectively; p is some fixed point on the meridian, the figure representing it as 36° north of the equator. Around p, a small circle is supposed to be described,

every point of which is $x°$ from p. In the figure, x is equal to 28°, hence the circle touches the north and south meridian at the points a and b, respectively at 8°, $=36°-28°$, and 64°, $=36°+28°$. It is evident that all possible lines of projection

5, *A* and *B*.



running from S to the small circle under consideration must be located upon the surface of an imaginary cone, with its apex at S and having a circular base touching the meridian at a and b. Such a cone is an *oblique cone*, and the plane $a\,b\,S$ is a symmetrical section through it. Continuing the line $S\,a$, laying off a distance $S\,b' = S\,b$, and joining $b\,b'$, a section $b\,b'$

through the cone will be an ellipse, but it is not necessary for the present discussion to demonstrate this point. A line joining the center of $b\ b'$ and S, passes through p, bisects the angle of the cone at S, and may be regarded as an axis of the cone. Let it now be imagined that the cone $b\ b'\ S$ revolves about the axis $S\ p$. Since the section $b\ b'$ is an ellipse, and an ellipse is a figure of binary symmetry, the surface of the cone during the first quarter of the revolution will depart somewhat from its original position, and then on continued turning it will approach more and more to its original position until a revolution of 180° has been accomplished, when the correspondence of the conical surfaces will be complete. After the revolution of 180°, the points a and b of the cone in its original position will be transferred to a' and b', respectively, and the section $a'\ b'$ must be a circle, because $a\ b$ was a circle. Figure 5, *A*, illustrates a most important feature of the stereographic projection; namely, *that no matter where p is located, if the pole of the projection is at S, all circular sections corresponding to a' b' are horizontal and parallel to the plane of the equator; hence, the lines of projection running from S to the circle a b or a' b' intersect the plane of the equator in a circle.* Proof of this feature of the stereographic projection is very simple. The angles β and β', figure 5, *A*, are equal, because they belong to the same cone before and after a revolution of 180° about the axis $S\ p$. Draw the line $a\ c$ parallel to $b'\ a'$, and β'' will equal β' because of the construction. The angle β (on the circumference of the circle at b) is measured by half the arc $S\ a$, and the angle β'' is measured by half the arc $S\ c$; therefore, since β and β'' are equal, the arcs $S\ a$ and $S\ c$ are equal. This being true $a\ c$ must be a chord, at right angles to a line joining the north and south poles, and hence *horizontal*, or parallel with the trace of the equator $X\ Y$. What holds good for a circle cutting the meridian at the points a and b, figure 5, A, holds good for a circle in any possible position on the sphere; hence, *all circles on the sphere will appear in stereographic projection as circles on the plane of the projection.*

Figure 5, *A*, further illustrates an interesting feature of conic sections; namely, that through an oblique cone circular sections are possible in two directions,—parallel to $a\ b$ and $a'\ b'$. All other sections are ellipses.

The construction of a small circle in stereographic projection corresponding to the problem of which, so to speak, an elevation has just been given in figure 5, *A*, becomes a very simple matter, and is illustrated in figure 5, *B*, case *I*. The divided circle here corresponds to the equator, and the north pole N is in the center. A north and south meridian would appear in the projection as a diameter of the circle, and the

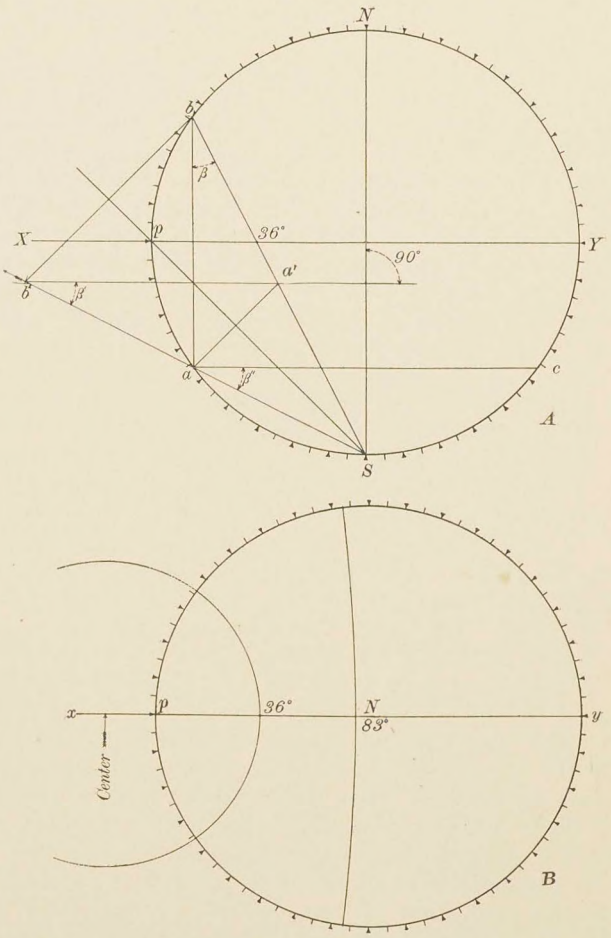
pole p, 36° from the equator, is readily located on a diameter by means of protractor No. I, p. 7. On just what diameter it should be placed would depend upon some determining factor—for example, the longitude of the meridian. The small circle which is to be projected is $x° = 28°$ from p; hence it would intersect the north and south meridian at points 8° and 64° from the equator, which points can be quickly located on the same diameter as p by means of protractor No. I. All that now remains to be done is to find the center point and construct the circle.

Another small circle in stereographic projection is illustrated by case II, figure 5, *B*. Here p' is some pole which may be located anywhere within the circle. Draw a diameter passing through p', bring the base line of protractor No. I to correspond with the diameter, and note the position of p'. In the case under consideration, p' was found to be 60° 20′ from N. The small circle described about p' is distant 45° from p'; hence it will touch the diameter at $15° 20' = 60° 20' - 45° 0'$, and at $105° 20' = 60° 20' + 45° 0'$. The two points 15° 20′ and 105° 20′ can be located on the diameter by means of scale No. 3, figure 3, and it then becomes an easy matter to find the center point and construct the circle. Numerous applications of the principles of the stereographic projection of small circles will appear during the course of this article.

The Stereographic Projection of Vertical Small Circles.—This is a problem deserving special consideration because of its very general application. The point p, figure 6, *A*, is located on the equator at the crossing of some meridian; hence the small circle described about it touches the north and south meridians, passing through p at points a and b, equally distant from p. The same demonstration that was employed on p. 9, figure 5, *A*, for illustrating that a small circle on the sphere is projected as a circle on the plane of the equator, holds good in the present case. Figures 5, *A*, and 6, *A*, are lettered alike, and the demonstration need not be repeated. In order to construct a vertical small circle in stereographic projection, figure 6, *B*, four points can be readily fixed upon. Two of these are the projection upon some diameter, $x\,y$, of the points a and b of the upper figure, the other two being points on the equator at the desired distance, $x°$, from p. To facilitate the projection of any desired vertical small circle, scale No. 2 of figure 3 has been constructed, from which the radius of the desired circle can be obtained. To construct, therefore, a small circle 36° from p, as represented by figure 6, *B*, draw a diameter $x\,y$ through p, and upon it, by means of the scale on the base line of protractor No. I, locate a point 36° from p. Then set a pair of dividers so that their points

will span the distance from 0° to 36° on scale No. 2, figure 3, find the center point on the diameter $x\ y$, and draw the small circle, which will intersect the divided circle at 36° from p. Owing to the character of the stereographic projection, the greater part of projected vertical circles will always lie outside the divided circle.

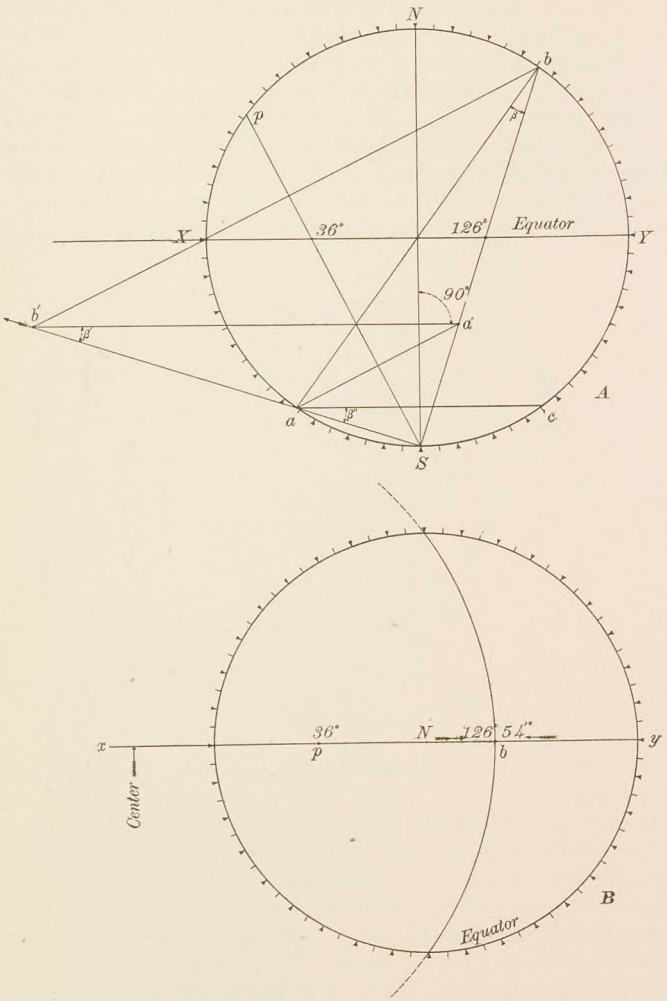
6, *A* and *B*.



Vertical circles nearly 90° from p have very long radii, and are best constructed by means of the curved ruler described on page 48. Thus, a vertical circle 83° from p, figure 6, *B*, must pass through two points 83° from p on the divided circle,

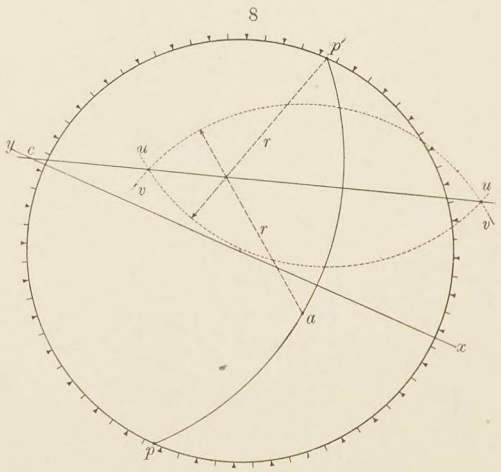
and also through the stereographically projected 83° point, as plotted on the diameter $x\ y$, by means of the graduation on the base line of protractor No. I.

7, *A* and *B*.



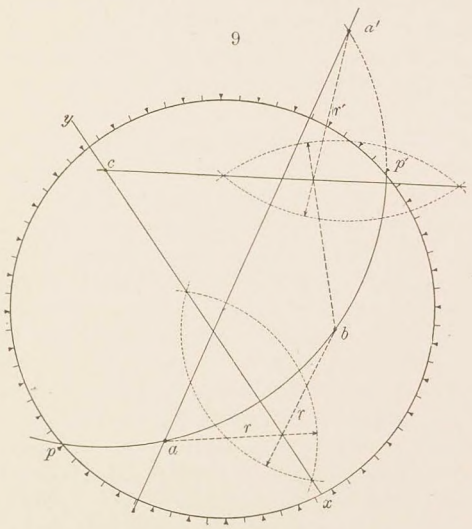
The Stereographic Projection of Great Circles.—What was true of a small circle described about some point p as a center, holds true also for a great circle intersecting a north and south meridian, figure 7, *A*, at points a and b, 90° from p. In the special case illustrated by the figure, p being 36° from the

equator, the line of projection from S to b crosses the plane of the equator at 126° (stereographically projected) from the left-hand end of the diameter XY, or 54° from the right-hand end, while the line of projection from S to a would intersect the plane of the equator far out beyond the divided circle, as indicated by the arrow, at a point which could be determined by scale No. 3, figure 3, a being 144° from N. All possible lines of projection from S to the great circle $a\ b$ are located on the surface of an imaginary oblique cone with its apex at S. Moreover, it could be proved, as was done in the case of the small circle illustrated by figure 5, A (the figures being lettered the same), that the intersection of the cone with the plane of the equator is in this case also a circle. The stereographic pro-



jection of a great circle is illustrated in figure 7, B. The projection of the point b, 54° from the equator on a north and south meridian, as shown by the upper figure, is quickly found on the diameter $x\ y$ by means of the graduation on the base line of protractor No. I. A circular arc must then be found passing through b, and intersecting the divided circle at antipodal points at right angles to the diameter $x\ y$. To facilitate the construction of such circular arcs, scale No. 1 of figure 3 has been constructed, which gives the radii of possible great circles. As the arcs of projected great circles approach N (the center of the divided circle) they become flatter; hence they are best constructed by means of the curved ruler described on page 48. As seen from scale No. 1, figure 3, the shortest radius of any stereographically projected great circle is equal to the radius of the divided circle.

To draw the projected arc of a great circle through two points, one of which, p, figure 8, is on the divided circle and the other, a, within the circle, is a simple matter. The circular arc must intersect the divided circle at p', antipodal to p, and its center must be on a line $x\ y$ intersecting the divided circle at 90° from p and p'. The center c may be found by a few trials with a pair of dividers, or it may be determined analytically as follows: With dividers opened up to some convenient distance construct two circular arcs $u\ u$ and $v\ v$, figure 8, having the same radius r, and draw a line through their intersections. The line thus drawn will be at right angles to the cen-



ter point of a line joining a and p', and will intersect the line $x\ y$ at c, which will be the center of the circular arc $p\ a\ p'$.

To draw the projected arc of a great circle through two points a and b, both of which are within the divided circle, the following principles may be used: That the great circle passing through a and b must also pass through points a' and b', antipodal, respectively, to a and b; also that the great circle must intersect the divided circle at antipodal points p and p'. If, therefore, there are two points, a and b, figure 9, anywhere within the circle, draw a diameter through one of them, a for example, and continue it beyond the circle. Apply the base line of protractor No. I to the diameter, determine the distance, in stereographically projected degrees, of a from the divided circle, and, making use of scale No. 3, figure 3, locate

a' just as many projected degrees beyond the divided circle as a is within it. Thus, as measured on a stereographically projected north and south meridian, a' is antipodal to a, and the problem of finding the center c and drawing a circular arc through a, b, and a', which is fully illustrated by the figure, is too simple to need more detailed explanation. In some cases it may prove easier to plot a line $x\ y$, as illustrated by figure 9, and find upon it the point c by trial, for only one circular arc can be found, which, passing through a and b, intersects the divided circle at antipodal points p and p'. If the two points within the circle are so located that the projected great circle passing through them has a very long radius, the curved ruler described on page 48 can be quickly adjusted so that a circular arc may be drawn passing through them and intersecting the divided circle at antipodal points.

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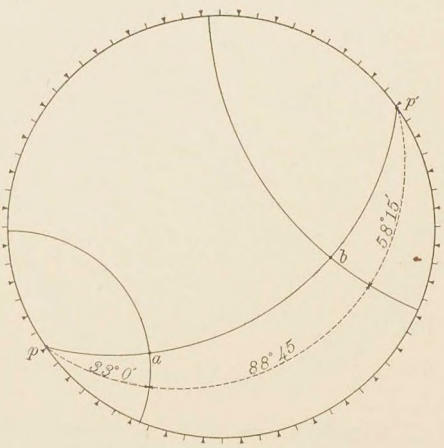

To measure the Angular Distance between any two points on a Stereographic Projection.—To measure the Side of a Spherical Triangle.—Let the two points a and b be anywhere within the divided circle, figure 10. Since the angular distance between any two points on a sphere is measured in degrees along the arc of a great circle, it is first necessary to construct a great circle passing through a and b, and thus locate the antipodal points p and p' on the divided circle. It is now possible to find some projected vertical small circle described about p, which passes through a and serves as a measure of the angular distance p to a; likewise a small circle described about p' and passing through b, which serves to measure the distance from p' to b. Knowing the angular distances p to a and p' to b, the distance a to b is readily determined. In fig-

ure 10, p to $a = 33°\ 0'$ and p' to $b = 58°\ 15'$; hence a to $b = 88°\ 45'$, or the supplement of the sum of the two angles.

Without the aid of the special protractor described in the next paragraph, it is a laborious task to find the two projected vertical circles passing through a and b, figure 10; though by making use of the protractor No. I and scale No. 2, figure 3, they can be plotted without great difficulty.

Stereographic Protractor No. II.—To facilitate measurements of the angular distances between points on a stereographic projection, or measurements of the sides of spherical triangles, a special protractor has been devised which is represented in plate I, without reduction, and will be designated as the *Stereographic Protractor*, or *Protractor No. II.* The essential features of this protractor are as follows: The circle of 14^{cm} diameter, divided into degrees, corresponds with the divided circle shown, much reduced, in figure 3. On one-half of the circle, the projected arcs of vertical small circles, p. 11, have been constructed for every degree. Since on a 14^{cm}-circle the stereographically projected circles are very near together, the even degrees have been represented by full lines and the odd degrees by dashed lines; also the arcs of every fifth and tenth degree are engraved somewhat heavier than the others. These details, however, are not essential, but simply serve to make the use of the protractor somewhat easier. On the other half of the circle only the arcs of every fifth and tenth degree are represented. This half of the protracter is really superfluous, but for approximate measurements it will at times be found more convenient than the other half where the arcs are crowded. For convenience the protractor is numbered in two directions, from 0° to 180°. Further, in order to have the protractor really practical, it should be printed or engraved on some transparent material; transparent celluloid has been found to satisfy every requirement.

To use the protractor in finding the distance from a to b, figure 10, for example, lay it (best with the printed side down) on the drawing, and bring the 0° and 180° points to correspond with the antipodal points p and p' on the divided circle, then note the distances p to a and p to b in degrees and fractions, and the difference between the two readings will equal the angular distance from a to b.

To measure from any given point p on the divided circle to a point a within the circle, it is not necessary to go through the operation of constructing the arc of a great circle through p and a, as represented in figure 8, p. 14; merely place the protractor with its 0° and 180° points on p and p', and then note on the protractor the projected arc which corresponds most nearly to a.

To measure the Angle made by the meeting of Two Great Circles.—To measure the Angle of a Spherical Triangle.— Just as the angle between two meridians at the north pole of a sphere is measured on the equator, so the angle between two great circles crossing at a point A, which may be anywhere within the divided circle, figure 11, is measured on the arc of a great circle at 90° from A. To make the measurement of the angle, draw a diameter of the divided circle through A, and, applying the base line of protractor No. I to the diameter, note the distance in projected degrees from N to A; then locate a point B just as many degrees from the divided circle as A is from N. A and B are thus 90° apart, and, making use of

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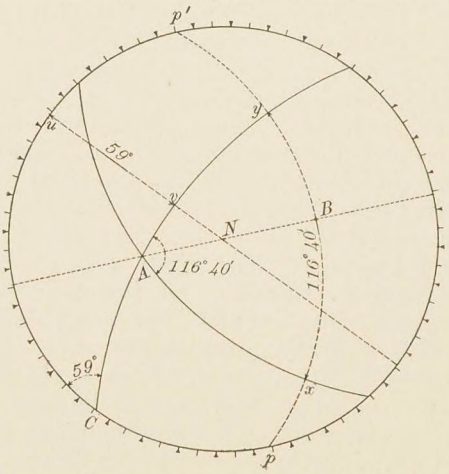

scale No. 1, figure 3, the arc of a great circle $p\ p'$, passing through B, can be easily drawn. All points on the great circle thus plotted, including the intersections x and y, are 90° from A, and the angle at A is equal to the distance in projected degrees between x and y, as measured with the stereographic protractor on the arc of the great circle $p\ p'$.

Provided an angle is located on the divided circle, as at C, all that is necessary to do is to draw a diameter of the divided circle at 90° from C, and measure the angle at C on the projected north and south meridian by means of the graduation on the base line of protractor No. I; for example, from u to $v = 59°$.

Another method of measuring the angles of spherical triangles depends upon a well known and interesting peculiarity of the stereographic projection, to which the writer's attention was called by Prof. G. P. Starkweather of Yale University; namely, *that the angle made by the crossing of two circles on*

a sphere is preserved in the stereographic projection of the circles. The method of measuring is illustrated by figure 12. ABC is a right spherical triangle, and to measure the angle B, draw through B a tangent tt'; the angle formed by tt' and the diameter CC' determines B. To draw accurately the tangent tt', locate two points a and b, equally distant from B, on the great circle ABA'. A line joining ab would be a chord of the circular arc ABA', and a line through B, parallel to ab, is a tangent to the circle. An ordinary protractor PP, preferably a transparent one,* centered at B by means of

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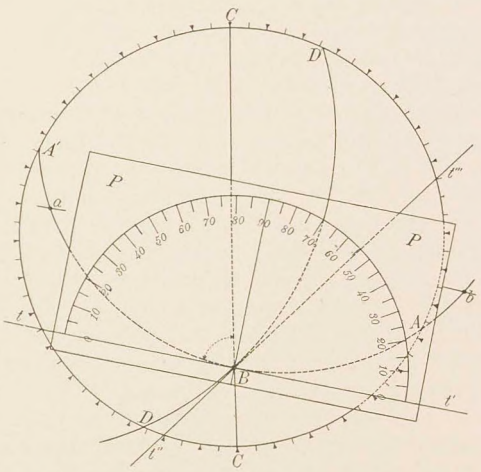
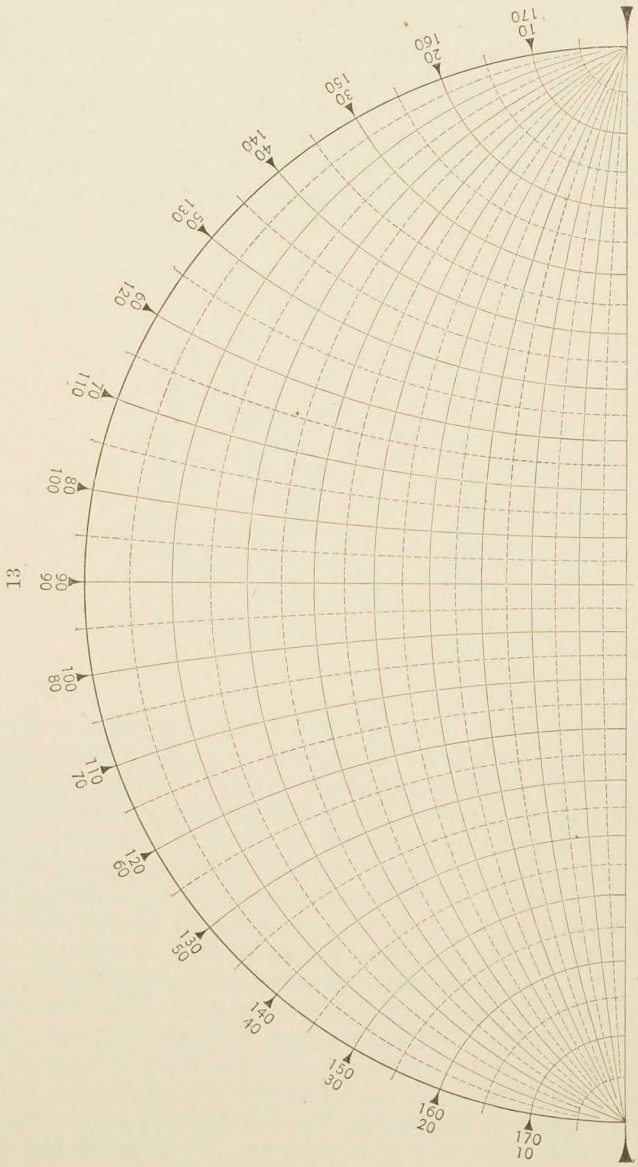


a needle, can be employed for measuring the angle. The 0°–0° line of the protractor is made to coincide with tt', and the angle (78° 15′ in the illustration) is determined from the graduation of the protractor. It is not necessary to construct the tangent tt' except when great accuracy is required; it is simply necessary to turn the protractor until the arc ABA' intersects the circular arc of the protractor at equal distances from its two 0° points when its 0°–0° line will be tangent to the arc ABA' at B. In case the spherical triangle is an oblique one, ABD, to measure B draw two tangents, tt' and $t''t'''$, then measure the angle between them. Whether it is easier to construct tangents and measure with an ordinary protractor, as illustrated by figure 12, or to construct the arc of a great circle at 90° from B, as illustrated by figure 11, and measure by means of protractor No. II, plate I, is a matter which must be determined by experience. It is far simpler to

* Protractor No. II, plate I, which is printed on transparent celluloid and is divided at the periphery into degrees, can be employed.

measure the angles A and D, figure 12, by means of the graduation on the base line of protractor No. I, page 7, than by constructing tangents and measuring the plane angles.



Supplementary Protractors.—Equipped with sheets containing the divided circle and scales, figure 3, and with protractors

I and II, figure 4 and plate I, practically all kinds of problems in spherical trigonometry can be plotted and solved by graphical methods. Two additional protractors, however, are so convenient that some space may be well devoted to them.

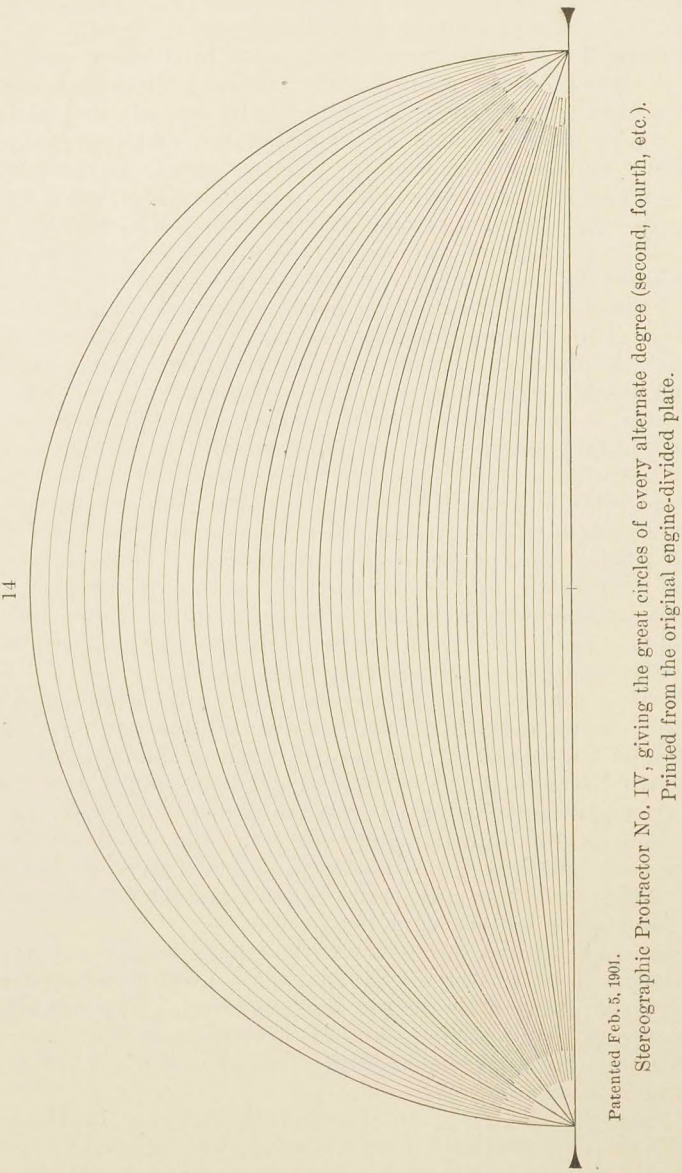
Protractor No. III, figure 13, is a combination of great and small circles on the same sheet. Every fifth degree only is given, the even degrees being indicated by full, and the odd degrees by dashed, lines. It is printed on transparent celluloid and is intended to be used for making an approximate measurement of the distance between any two points on a stereographic projection. At the center, there is a small hole, and if a sheet upon which a stereographic projection has been made is pierced at its center, the protractor can be quickly centered upon the sheet by means of a needle passed through the two holes. Thus centered, the protractor is turned until the two points, the distance between which is to be measured, fall either on an arc of a great circle or at proportionately equal distances between the arcs of two great circles. The distances of the two points from either zero point of the protractor is then noted, as indicated in degrees by the stereographically projected small circles, and the difference gives the distance between the points. Generally speaking, it is not essential to know upon what great circle the two points are located. They must be on some great circle, and the 0° and 180° points of the protractor locate approximately where this great circle intersects the divided circle. It is from the antipodal points thus ascertained that the readings of the protractor are made.

Given a chart or map in stereographic projection and a protractor, as just described, less than two minutes are required to shift the protractor to the proper position and measure the distance between any two points. Measurements thus made with a protractor graduated only to every fifth degree can be regarded as but approximately correct. They will seldom be more than a degree from the truth, and will average less than half a degree. For quick and approximate work, therefore, this protractor will be very serviceable.

The great circles for every fifth degree might well appear on that half of protractor No. II, plate I, on which the small circles for every fifth degree are engraved. The advantage, however, of having the two kinds of circles on the same protractor was not appreciated until after the plate for protractor No. II had been engraved.

Protractor No. IV.—In this protractor, figure 14, the great circles of every alternate degree (second, fourth, etc.) are represented. On the small scale adopted, it would bring the arcs too close together to represent every degree. The arcs are not numbered, but those corresponding to every tenth degree are represented by lines heavier than the rest, so that they may be

readily caught by the eye. The protractor is printed on transparent celluloid. At the center there is a small hole, and at



the ends of the semicircle the celluloid is cut away to the exact line of the diameter. The chief use of this protractor is to find the great circle which passes through any two given points

of a stereographic projection; more especially, however, to locate the antipodal points where the great circle crosses the divided circle. To perform this operation, the protractor is centered by means of a needle over a sheet upon which a stereographic projection has been made. It is then revolved until any two points of the projection are on, or proportionately distant from, the same great circle, when, by means of a pencil, marks are made on the drawing where the base line of the protractor crosses the divided circle. From the antipodal points thus located, measurements to the two points within the circle can be made by means of protractor No. II. In the majority of cases it is not necessary to draw the arc of a great circle passing through two points within the circle. To measure the distance between them, however, it is necessary to locate the antipodal points on the engraved circle. On page 15, figure 9, it was shown how a great circle passing through any two points within the circle of a stereographic projection may be plotted. Protractor No. IV furnishes a quick, though perhaps not quite so accurate a method of accomplishing the same result.

The Engraving of the Scales and Protractors.—It seems best to give a brief account of the methods employed in making the scales and protractors. The graduation of scale No. 3, figure 3, repeated in part on the base line of protractor No. I, figure 4, was calculated to a fraction of a millimeter by means of a simple tangent relation; for example, the distance of the 20° line from the center equals the radius of the circle multiplied by the natural tangent of 10°, page 4. The method of preparing scale No. 1, figure 3, may be illustrated by referring to figure 7, *A*, page 13. The distances (from the center) of the stereographically projected 126° point to the right and of the 144° point to the left, as determined by scale No. 3, are added, and their sum divided by two gives the radius of the great circle under consideration. The radius of the great circle 54° from the periphery also equals the distance from the stereographically projected 54° point to the right to the stereographically projected 108° (twice 54°) point from N, to the left of the center, and a like relation holds good throughout. Figure 6, *A*, page 12, illustrates how scale No. 2, figure 3, was derived from scale No. 3. From the distance (from the center) of the stereographically projected 126° point, that of the 36° point was subtracted, and the difference divided by two gives the radius of the small circle under consideration. This radius is also equal to the distance from the center to the stereographically projected 72° (twice 36°) point from N, a like relation holding good throughout.

Scale No. 4 is not employed in the stereographic projection, but is used in crystallography. The data supplied for making

scales 1, 2, and 3 was employed in the construction of the protractors. Scales and protractors were laid out and engraved by means of a dividing engine. The plates were made by the so-called wax process. In this process, a metal plate, covered with a wax preparation, is employed, and the engraving tools cut through the wax to the metal. Engine work on wax plates should leave the engraver's hands practically perfect. An electrotype of the plates is next taken, and the copper depositing in the grooves made by the engraver's tools furnishes the relief from which impressions are taken. Finally, in order to prepare the electrotype for printing, the copper, stripped from the plate, is strengthened by casting some easily fusible metal upon the back of the electrotype. This casting causes a slight shrinkage of the plates, amounting to about $0{\cdot}5^{mm}$ for a distance of 140^{mm} (the diameter of the protractors). It has been ascertained that the shrinkage is practically uniform and proportional in all the plates, so that no appreciable errors can arise from using engravings of this kind. Still, the plates are not absolutely perfect.

Copper or steel engravings would be preferable to engravings made by the wax process, but both the plates and impressions therefrom would be more expensive. If it is thought best to have another set of scales and protractors on a larger scale, the desirability of having them engraved on steel or copper should be carefully considered. Even impressions made from steel engravings might not be absolutely perfect, for paper changes somewhat under varying climatic conditions, and expansion and contraction are not always alike in every direction.

Another thing to be considered is that celluloid shrinks, and protractors printed from accurately engraved plates may become inaccurate in a few months. Except for its tendency to shrink, however, celluloid is exactly the kind of transparent material upon which to print the protractors, and a very simple correction serves to offset the slight shrinkage. For example, some impressions of protractor No. II, printed on newly purchased celluloid five months prior to the time of writing, have shrunk so that with a zero point of the protractor matched exactly at one end of a diameter of the divided circle, figure 3, the 180° point falls one-half a degree short of coincidence with the opposite end of the diameter. In other words, a distance of 179° 30′, stereographically projected upon a diameter of one of the engraved sheets, figure 3, is measured as 180° by means of the protractor; hence, from measurements made by the protractor, 5′ should be deducted for every 30°, equivalent to 30′ for 180°. It is believed that, in time, celluloid will become seasoned, so to speak, and that it will cease to shrink. Accordingly a rather large stock of this material has been purchased with the hope that in the course of a year or two it will be possible to print protractors which will not change.

Taking into consideration all imperfections of the plates, especially those due to shrinkage in casting, errors arising therefrom are quite insignificant when compared with the errors involved in plotting points and constructing arcs when the construction is based upon so small a scale as a 14^{cm} circle.

Some Results obtained from the Solution of Spherical Triangles by Graphical Methods.—In order to test the accuracy of the methods set forth in the foregoing pages, two right and five oblique-angled triangles were plotted and solved graphically. None of the parts of the triangles selected were given to exact degrees, so at the very beginning the points had to be approximately located between the degree graduations of the 14^{cm} divided circle and the scales derived therefrom.

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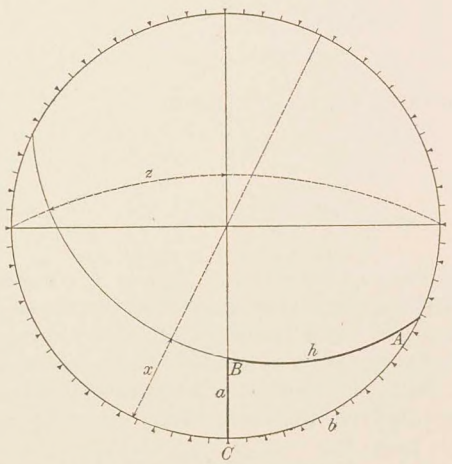

Case 1.—Given two sides, $a = 26° 34'$ and $b = 63° 26'$, of a right triangle, figure 15. At C construct a diameter, and, by means of protractor No. I, plot the side a, 26° 34′. Lay off the side b, 63° 26′, on the divided circle, and then construct a great circle passing through the points thus located, page 15. The angles A and B and the side h of the triangle are thus plotted. A is now measured on the diameter x (stereographically projected great circle) at 90° from A, by means of protractor No. I. B is measured on the great circle z, at 90° from B, by means of protractor No. II, page 18. The side h is measured by protractor No. II. The results of the work are as follows:

	Calculated.	Plotted.	Error.
A =	29° 12′	29° 10′	2′
B =	77 24	77 30	6
h =	66 25	66 25	0

Case 2.—Given an angle $A = 24° 9'$ and a side $h = 70° 14'$ of a right triangle, figure 16. On the diameter x, locate the point p 24° 9′ from the divided circle, using protractor No. I; then making use of scale No. 1, figure 3, construct the great circle $A\,p$. Likewise on the diameter y locate p' 70° 14′ from

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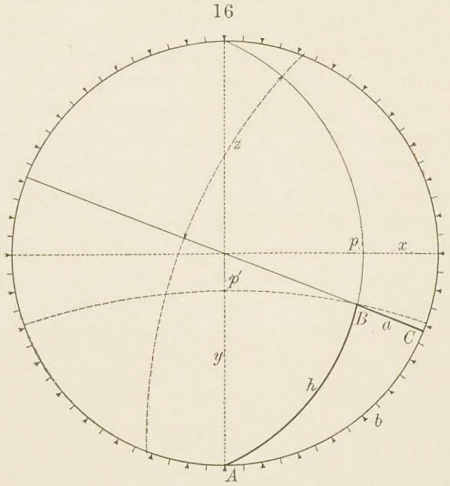

A, and making use of scale No. 2, figure 3, construct the small circle which intersects the great circle previously constructed at B. Through B draw a diameter, thus locating the right angle C, and completing the triangle. The side a is measured by protractor No. I, the side b by the graduation of the divided circle, and the angle B on the great circle z, by means of protractor No. II, page 18.

The results are as follows:

	Calculated.	Plotted.	Error.
a =	22° 39′	22° 40	1′
b =	68 30	68 25	5
B =	81 23	81 25	2

Case 3.—Given the three sides of an oblique triangle, $a = 94° 26'$, $b = 78° 42'$, and $c = 72° 36'$, figure 17. On the divided circle, lay off the side a, and thus locate the angles B and C. On the diameters x and y locate the points p and p', respectively 78° 42′ and 72° 36′ from C and B, using protractor No. I. Now making use of scale No. 2, figure 3, construct small circles through p and p', and their intersection locates the angle A. Construct the great circles through C and A, and B and A, and the sides a, b, and c, of the triangle are plotted. The angles B and C are measured on the diameter x' and y', at 90° from B and C, respectively, using protractor

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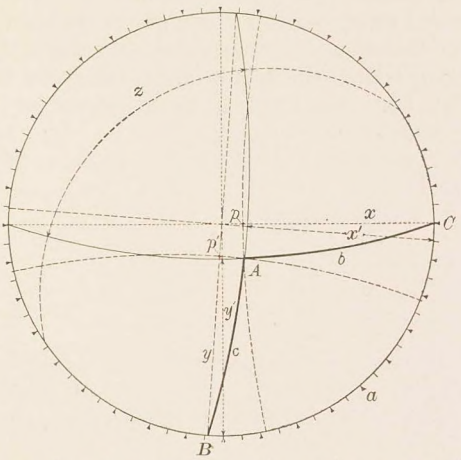

No. I. The angle A is measured on the great circle z, using protractor No. II, page 18.

The results are as follows:

	Calculated.	Plotted.	Error.
A =	98° 21′	98° 30′	9′
B =	76 47	76 40	7
C =	71 16	71 15	1

Case 4.—Given the two sides and the included angle of an oblique triangle, $a = 31° 25'$, $b = 88° 53'$, and C = 97° 13′, figure 18. To plot the angle C, on the diameter x locate the point p 97° 13′ from the divided circle, and construct the great circle $C\,p$. To plot the side a and thus locate B, on

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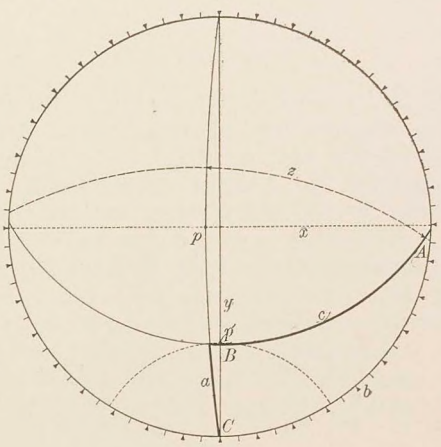

the diameter y locate the point p' 31° 25′ from C, and construct the small circle with radius given by scale No. 2, figure 3. Lay off b on the divided circle, and draw the great circle AB, thus completing the triangle. The side c is measured by protractor No. II. The angle A is measured on a diameter 90° from A, by means of protractor No. I. The supplement of the angle B is measured on the great circle z, using protractor No. II, page 18.

The results are as follows:

	Calculated.	Plotted.	Error.
A =	31° 11′	31° 5′	6′
B =	83 15	83 0	15
c =	92 48	92 50	2

Case 5.—Given two angles and the included side of an oblique triangle, $A = 59° 3'$, $B = 53° 34'$, and $c = 75° 23'$, figure 19. Lay off the side c on the divided circle. Plot the

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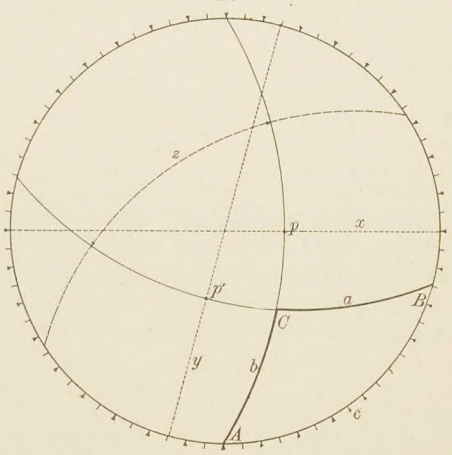

angles A and B by locating the points p and p', on the diameters x and y, respectively, at 90° from A and B, the point p being 59° 3′ and the point p' 53° 34′ from the divided circle. Using scale No. 1, figure 3, construct the great circles through A and p, and B and p', thus locating C and completing the triangle. The sides a and b are measured by protractor No. II, and the angle C also by protractor No. II on the great circle z, page 18.

The results are as follows:

	Calculated.	Plotted.	Error.
a =	56° 50′	56° 45′	5′
b =	51 45	51 40	5
C =	97 33	97 35	2

Case 6.—Given two sides and the angle opposite one of them of an oblique triangle, $a = 56° 7'$, $b = 40° 8'$, and $A = 93° 39'$, figure 20. This problem has but one solution when a is greater than b, as in the present case. Lay off b on the divided

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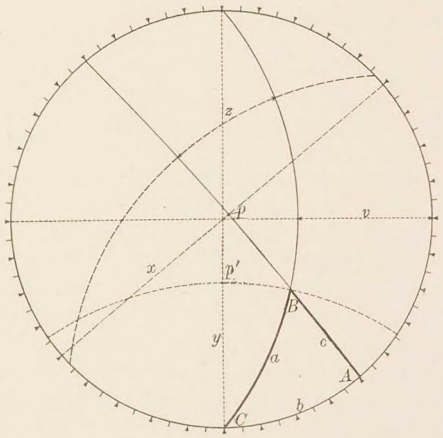

circle. On the diameter x, 90° from A, locate p 93° 39′ from the divided circle, and draw the great circle Ap. On the diameter y locate p' 56° 7′ from C, and construct the small circle which determines B. Through C and B draw a great circle, which completes the triangle. The angle C is measured on the diameter v, at 90° from C. The angle B is measured on the great circle z by protractor No. II, page 18. The side c is measured by protractor No. II.

The results are as follows:

	Calculated.	Plotted.	Error.
B =	50° 47′	50° 55′	8′
C =	50 53	51 0	7
c =	40 12	40 25	13

Case 7.—Given two sides and an angle opposite one of them of an oblique triangle, $a = 63° 20'$, $b = 73° 8'$, and $A = 55° 10'$, figure 21. This problem is similar to the foregoing in the parts given, but has two solutions when a is less than b. On the diameter x, 90° from A, locate p 55° 10′ from the divided circle, and draw the great circle Ap. On the diameter y locate p' 63° 20′ from C, and construct the small circle which intersects the great circle Ap, previously drawn, at two points, B and B'. Thus two triangles are plotted, ABC and $AB'C$, determining the two solutions of this problem. The acute and obtuse angles at C are measured on the diameter v, at 90°

from C. The obtuse angle B is measured on the great circle z by protractor No. II, page 18. Likewise the acute angle B' is measured on the great circle z', the portion between the arrow points giving the supplement of the angle. The sides c and c', from A to B and from A to B', are measured by protractor No. II.

The results are as follows:

	Calculated.	Plotted.	Error.		Calculated.	Plotted.	Error.
B =	117° 41′	117° 37′	4′	B' =	62° 19′	62° 20′	1
C =	20 17	20 11	6	C =	103 48	103 55	7
c =	18 43	18 55	12	c =	116 1	115 55	6

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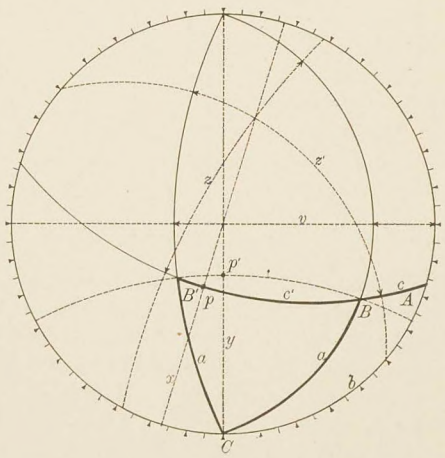

The solutions of the seven problems just given were made without knowledge of the calculated values; furthermore, the results are not a selection of best values, obtained from a number of solutions of the several problems. The results have been given exactly as they were obtained, and hence they serve to illustrate the probable degree of accuracy which may be secured. In solving the two right triangles the greatest error was 6′, the average error less than 3′. In solving the five oblique triangles the greatest error was 15′, the average 7′. It is evident that by increasing the size of the projection, and using more accurately engraved plates, the errors could be materially lessened.

One possible case has not been included in the foregoing list; namely, having three angles of a triangle given. Such a problem may be solved graphically, but, having no sides given, the problem is complicated. It may be done somewhat as follows: Referring to figure 19, the great circle Ap, determin-

ing the angle A, is first plotted. Then, taking protractor No. IV, page 22, the great circle corresponding most nearly to B is selected, and the protractor, centered on the drawing, is turned until the great circles $A\ p$ of the drawing and $B\ p'$ of the protractor cross at an angle approximating to C, which is told by means of an ordinary protractor, page 19. Thus an approximate solution may be quickly obtained.

Some Practical Applications of the Stereographic Projection.—Some mathematicians may contend, and justly, that the seven problems previously presented might be solved easier by numerical calculations than by plotting; and that by using four- or five-place logarithm tables the results would be correct to minutes, while those obtained by plotting are only approximately correct. In spite of the truth of this, graphical methods still have their advantages. Most persons who have occasion to make calculations, the writer included, use formulas and tables, as a rule, wholly in a mechanical way. With graphical methods, on the other hand, formulas and tables are not needed, and every operation is clearly understood, for graphical methods can scarcely be applied otherwise. In the majority of cases, numerical calculations are laborious, while graphical solutions appeal to one like pictures which, to a certain extent, tell their own story. Although it may be known that for some problems there are two solutions, it is safe to assume that of those persons who are accustomed to solve spherical triangles by the use of formulas, only a few could satisfactorily explain why two solutions apply to the conditions shown in figure 21 where a is less than b, while only one solution is possible when a is greater than b, figure 20. The two figures, however, being accurately constructed, not only show the point in question clearly, but present the problems in such a manner that desired parts can be measured.

Most persons have never studied spherical trigonometry, and those who have studied it usually regard the solution of a spherical triangle as at least a laborious, if not a difficult, matter. This is especially true of those who for a long time have not made numerical calculations, and who have thus lost their familiarity with the use of formulas and of logarithmic tables. Perhaps no one can appreciate the difficulties presented by the subject more than one who has had occasion to teach crystallography to advanced students. It is generally the case that students desiring to take up crystallography (chemists especially) have not made numerical calculations, other than very simple arithmetical ones, for a number of years. As a consequence, the problems in spherical trigonometry are regarded as a great trial. Mistakes of all kinds arise, and the only way to make absolutely sure of one's work is to check, either by

means of duplicate calculations or by applying formulas for checking. Professor Edward S. Dana, in preparing his "System of Mineralogy," had all the angles in the book recalculated, using as far as possible the fundamental angles of the original investigators. He has told the writer that the number of errors he detected was astonishing. Some of the errors had been made even before the axial relations of the crystals were determined, and whole tables of angles were, as a consequence, vitiated. Other mistakes had been made early in the century, and had been copied into all the leading text-books of mineralogy and crystallography.

By making use of the graphical stereographic methods, triangles can be plotted practically in exact proportions, and, by measuring the unknown parts, a check upon the results of numerical calculations can be made. The importance of having some simple method of checking can not be overestimated. In the matter of making numerical calculations persons differ greatly. Some have great facility and seldom make mistakes; others do the work with difficulty, especially, it would seem, if, like the writer, they are only occasionally called upon to make calculations. Some check, therefore, carried out graphically, is a great saving of nervous energy. In the writer's short experience in using graphical methods as applied to the stereographic projection, numerous errors in numerical calculations have been detected. At times the error has not amounted to 1°, yet it was evident that some mistake had been made. Then, too, in following out the graphical methods, a map or chart is prepared, which in crystallography, as doubtless also in other departments of science, is of importance.

In determining geographical distances, the stereographic projection possesses very great advantages. In physical geography, for instance, it is important to find the distance between two points on the earth's surface in order to determine the velocity of wind and ocean currents, seismic waves, tidal waves, etc. In navigation, at least in teaching it from an elementary standpoint, the distance between two points of given longitude and latitude must be determined. To indicate how easily this may be done, the problem of finding the distances of New York and Rio de Janeiro from Queenstown will be presented. The approximate longitudes and latitudes of the places, as taken from an atlas, are as follows:

Queenstown.	New York.	Rio de Janeiro.
8° 15′ W.	74° 0′ W.	43° 0′ W.
51° 50′ N.	40° 40′ N.	23° 0′ S.

In equatorial stereographic projection, figure 22, the meridian of Greenwich is drawn from the center to 0°. Queenstown is

located on the meridian 8° 15′ W., and by means of protractor No. I, at 51° 50′ N. In a similar manner, New York is located.

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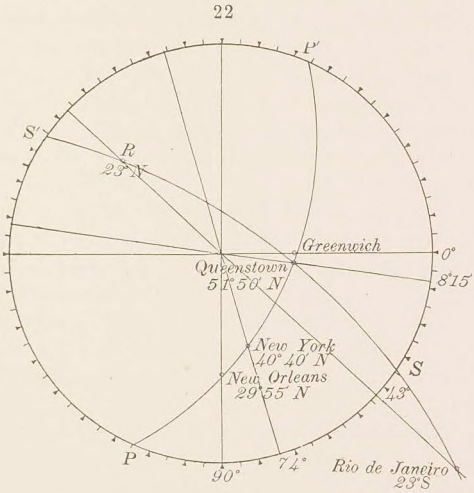
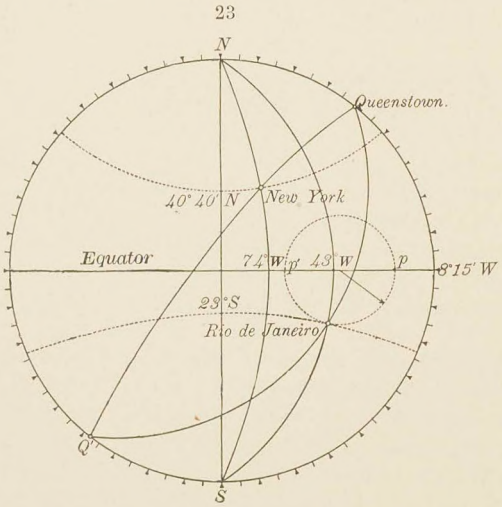


Rio de Janeiro being south of the equator falls beyond the engraved circle on the meridian 43° W., but can be easily located by means of scale No. 3, figure 3. Instead of locating Rio de Janeiro, however, beyond the engraved circle, its antipodal point R can be located on the same meridian, at 23° N. of the equator. Using now protractor No. IV, the points P and P', and S and S' are located, where the great circles passing through Queenstown and New York, and Queenstown and Rio de Janeiro cut the equator; then using protractor No. II the distances are measured. From Queenstown to New York the distance, as plotted, was found to be 45° 23′, calculated 45° 11′. From Queenstown to R the distance as plotted was 99° 20′; hence to Rio de Janeiro is the supplement of this value, 80° 40′; while by calculation it is 80° 47′. The data for the calculations were the longitudes and latitudes as previously given.

An advantage is gained by using a meridian rather than an equatorial projection, since the method of measuring is simpler, and the plotting is done near the periphery of the divided circle, where the distances between the stereographically projected degrees are largest. Figure 23 illustrates this method. The divided circle represents the meridian passing through Queenstown, and a mark at 51° 50′ locates the place. On the stereographically projected equator, the intersections of the meridians of New York and Rio de Janeiro are plotted by means of the graduation on the base line of protractor No. I. The two meridians 74° and 43° West of Greenwich are respec-

tively 65° 45′ and 34° 45′ West of the meridian of Queenstown. By taking the radii from scale No. 1, figure 3, the two meridians are quickly drawn. New York, being in latitude 40° 40′ N., is 49° 20′ south of the pole. Its position is determined by locating the point 40° 40′ N. on the central meridian, and constructing the small circle with radius 49° 20′ taken from scale No. 2, figure 3. In a similar manner, Rio de Janeiro is located. The figure also indicates another method of locat-



ing Rio de Janeiro. On the equator locate the points p and p' at 23° from the crossing of the meridian of Rio de Janeiro. The intersection of the meridian being 34° 45′ (43° — 8° 15′) from the divided circle, the points p and p' are respectively 11° 45′ and 57° 45′ from the divided circle. Now by finding the center and constructing the small circle, Rio de Janeiro is located. To measure the distances, match the zero point of protractor No. II with Queenstown, swing the protractor so that its center corresponds with that of the plate, and note the position of the points plotted. Thus plotted, the distances from Queenstown to New York and Rio de Janeiro were found to be respectively 45° 15′ and 80° 52′, calculated 45° 11′ and 80° 47′. Plotted in this way it is seldom that the error exceeds 6′; the average error is less than 4′. It is a decided advantage to be able to determine any distance by one reading of the protractor, rather than by two readings and a subtraction, as illustrated by figure 22.

It may be seen from the foregoing demonstrations that if there were maps of the northern and southern, and eastern and

western hemispheres accurately plotted in stereographic projection, and transparent protractors constructed on the same scale, such measurements could be made by simply shifting the protractors and noting the angles. Maps of this kind would be very serviceable, and the writer believes that they should and will be made. The most important continental features (promontories, mouths of rivers, lakes, etc.), the islands, and the principal ports and inland cities could be located with great accuracy on a map of 30^{cm} (nearly one foot) diameter, and it ought to be possible to measure distances between any two points within two or three minutes (two or three nautical miles) of the truth; while the maximum error ought not to exceed ten minutes. If there is an error of judgment in the foregoing statement, it favors greater, rather than less, exactness, for on the 14^{cm} circles a degree of accuracy can be obtained almost equal to that just expressed.

Without doubt geographers and physical geologists would find many uses for sheets printed from accurately engraved plates giving the meridians and parallels, in both equatorial and meridian stereographic projection, plates II and III. On such plates, points of given longitude and latitude could be quite accurately located (within half a degree), the outlines of continents sketched, wind and ocean currents noted, etc.; and, provided with protractor No. III, measurements sufficiently accurate for most purposes could be made in a very few minutes. The writer has not employed such sheets in crystallography,* but has had the plates prepared to conform with the protractors and scales described in this article, believing that they will prove very useful.

It is at times desired to shift a stereographic projection so as to bring some special point or pole to the center. By making use of scale No. 3, figure 3, this can be easily accomplished. Plate IV represents a projection thus shifted so as to bring longitude 75°, latitude 40° (practically the location of New York city), to the center. The north pole is shifted 50° from the center, and the equator is an arc of a great circle crossing the vertical diameter 50° from the divided circle. Cutting off scale No. 3, figure 3, from one of the engraved sheets, and matching it along the vertical diameter, the stereographically projected points 10°, 20°, etc., are laid off in both directions from the shifted north pole, and about the respective center points the parallels of latitude (small circles) are drawn. That parallel which is located by actual measurement exactly halfway between the stereographically projected north and south

* Impressions of an equatorial stereographic projection, essentially like plate II, have been recommended by Fedorow, especially for use with the two-circle or universal goniometer. Zeitschr. für. Kryst., xxxii, p. 446, 1900.

poles (the parallel 40° South, plate IV) has an infinite radius, and appears as a straight line in the projection. Upon this straight line are located the centers of all the stereographically projected meridians, for they are all arcs of circles running through the north and south poles, hence having their centers on a line crossing the middle point of the N. S. diameter at 90°. The points of intersection of the meridians with the equator may be found by drawing an arc, with a radius like that of the equator, on an impression of protractor No. II, plate I. Then, by means of dividers, the points 5°, 15°, 25°, etc., from the 90° line of the protractor are transferred to the shifted equator. It took very little time to construct the parallels and meridians of plate IV, and, considering the size of the projection, they are accurately drawn. The outlines of the continents were sketched upon the chart by Mr. H. H. Robinson of Yale University.

Professor Andrew W. Phillips of Yale University has devised a machine consisting of jointed rods, by means of which the pole of any stereographic projection can be shifted to any desired position. Thus an equatorial projection, the easiest of all to make, can be transformed into a meridian projection, or into one like plate IV, having some desired point at the center. This machine was exhibited in 1884 before the British Association for the Advancement of Science, at their summer meeting in Montreal,* and is described by Professor Phillips in his geometry.†

Map Projection.—The method of projection almost universally employed by geographers for representing hemispherical surfaces is the so-called *Globular Projection*, invented in 1660 by the Italian Nicolosi.‡ In this method the equator is divided into equal parts, and the meridians are circular arcs uniting these points with the poles; the parallels are likewise circular arcs, dividing the extreme and central meridians into equal parts. Figure 24 shows the meridians and parallels in globular projection. Compare this figure with the stereographic projection of the meridians and parallels, plate III, and a marked difference is at once apparent. The stereographic projection is correct in every particular, the parallels intersect the meridians at right angles, as on a globe, and, as has been shown, distances and directions can be accurately measured and plotted on such a projection. In the globular representation, on the other hand, nothing is correct except the graduation of the outer circle and the directions of the two diameters; distances and directions can be neither measured nor plotted.

* Report of the British Association for the Advancement of Science, 1884, p. 649.

† Phillips and Fisher's Geometry, 1899, p. 510.

‡ Germain, Traité de Projections des Cartes Géographique, p. 127, Paris, about 1865.

Strictly speaking, figure 24 is not a projection, for it does not correspond point for point with the surface of a sphere, according to some fixed law of projection. It is simply an arbitrary distribution of a series of curves. The only excuse for its continued use in map construction is that, having the distances on the equator and central meridian equally spaced, land and water areas are more uniformly distributed than in the stereographic

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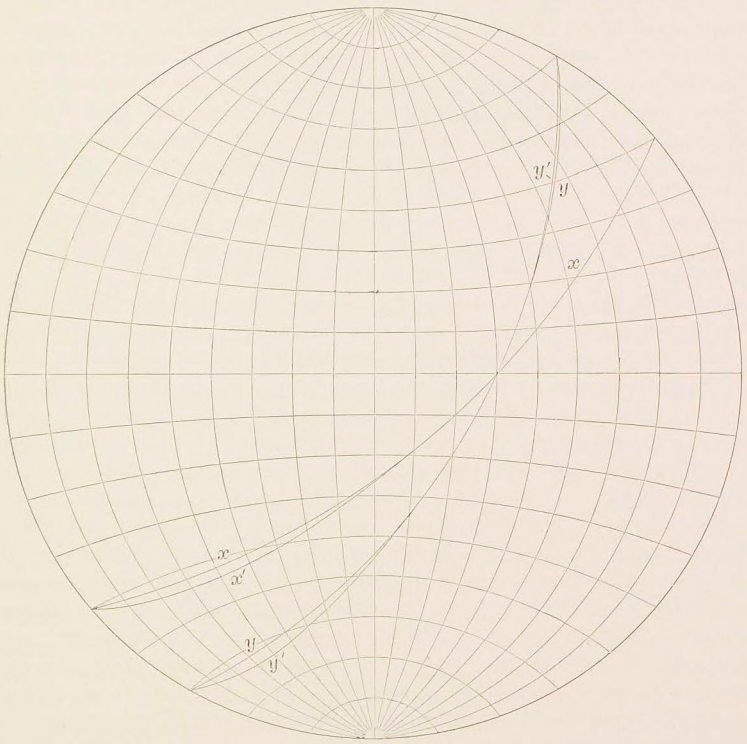

projection. In order to show still another defect of the globular representation, two circular arcs, x and y, have been drawn on figure 24, one running from 40° N. on the periphery to 60° W. on the equator, the other from 60° N. to 60° W. Circular arcs drawn from the same locations on plate III would represent stereographically projected great circles, agreeing in their intersections with the meridians and parallels point for point with corresponding great circles drawn on a sphere. Not so with the globular representation, however. The two great circles under consideration, if accurately plotted on the globular chart from their actual intersections with the meridians and parallels,

would appear not as circular arcs, but as the irregular curves x' and y', figure 24, showing marked deviation from circular arcs in the lower left-hand portions of the chart.

It is impossible to represent the areal relations of a hemisphere upon a plane without sacrificing some features. In the stereographic projection, plates II and III, distances between the meridians and parallels become smaller as they approach the center. Distances and areas on stereographic maps must therefore be magnified in proportion as the distances between the meridians and parallels become smaller. The gradual contraction of areas, as the center of a stereographically projected hemisphere is approached, is not altogether a drawback, for it should be part of a person's education to understand that, in making a map on a flat surface, some contraction or magnification of areas must appear on certain portions of the map. Doubtless most geographical relations can be appreciated best by beginners by studying a sphere or globe. Serious difficulties, however, are encountered in making accurate drawings and measurements on a spherical surface; hence to be able to plot all the relations of a sphere easily, quickly, and accurately on a flat surface is a great advantage, an advantage, moreover, which the stereographic projection alone possesses.

It would seem as though the distorted and inaccurate globular representation, now universally employed by geographers, should give place to the accurate stereographic projection. It is safe to assume that few teachers in our academies and high-schools have exact ideas concerning the kinds of projection employed in map construction. By making use of comparatively simple wire models* it should be possible to give not alone to teachers, but to scholars of from twelve to sixteen years of age, a sufficient knowledge of the essential features of the stereographic projection to enable them to appreciate the meaning and significance of meridians and parallels as projected on a flat surface, plates II and III.

If scholars were supplied with stereographic charts, corresponding to plates II and III, and were taught to locate places from their longitudes and latitudes, the more skillful of them, at least, would soon be able to construct quite accurate maps, better than those now existing in our school geographies, and

* The writer has in mind models such as are used in teaching crystallography. Wire circles could be arranged and soldered so as to represent meridians, parallels, and arcs of circles in any desired position. It does not take many wire circles to give to such models the effect of a sphere. By running wires or threads from certain fixed points on the circles to the south pole, for example, the fundamental conception of the stereographic projection,—*the projection to a pole on the surface of a sphere*—can be demonstrated. Where the wires or threads intersect the plane of the equator, determine the position of the points in stereographic projection. Great and small circles could be thus projected, and if properly worked out with not too much detail, the models would be very effective.

ones which would be *correct* within certain limits, depending upon the size of the projection and the skill with which the drawing and plotting were done. Map-drawing is generally a feature of grammar-school education, and by making use of printed charts, showing stereographically projected meridians and parallels, it would be no more difficult to construct an accurate map than a mere sketch, and probably it would be easier. It would be at least more profitable and instructive.

Having demonstrated on a globe that the shortest distance between two points is along the arc of a great circle, the use of protractor No. IV, figure 14, could be explained, and by turning the protractor, the great circle passing through any two points could be easily and quickly found. It is probable that by making use of a suitable wire model, the principles underlying the construction of protractors II and III could be made clear. In any event, an intelligent person would soon learn to turn protractors II and III to the right position, and determine in degrees the distance between two stereographically projected points.

Students come to the universities with altogether too little knowledge of how to do things accurately, and the lack of proper training in this respect is a serious defect in their education. By applying to geography the mathematically correct principles of the stereographic projection, it would be possible to inculcate into pupils of high-school, possibly also of grammar-school, age, methods of absolute accuracy as pertaining to map construction. The essential features of the projection are so simple, that, if properly presented with the aid of a few models and diagrams, it should be possible to teach comparatively young pupils how to construct maps intelligently and to make geodetic measurements accurately. It seems to the writer that training of such a nature would be most beneficial, not alone because it is important to know how maps are constructed and geodetic measurements are made, but, in a broader sense, because of the advantages derived from that kind of training which teaches pupils how to do things and to do them correctly.

This opportunity will be taken to present a brief demonstration showing the advantages of the stereographic over the ordinary methods of map construction. In maps comprising small areas, it is possible to measure the distance between two points, in a straight line across the map, with considerable accuracy. Not so, however, for long, transcontinental distances, as such must be measured with reference to the curvature of the earth along arcs of great circles. Most persons have no means of determining the distance between distant points other than the very crude, and necessarily incorrect

method of applying a scale of miles in a straight-away direction across a map. For the sake of demonstration, the distance between New York, 74° W., 40° 40′ N., and New Orleans, 90° 0′ W., 29° 55′ N., was chosen. Making use of atlases, only those maps can be employed on which both New York and New Orleans appear on the same sheet, therefore maps of the whole United States must generally be used. On a number of such maps, varying in width from 14 to 26 inches, and accompanied in each case by a scale of miles, the following results were obtained: 1170, 1180, 1185, 1190, 1200, and 1210 statute miles. On all the maps, the position of the two places with reference to the meridians and parallels corresponded closely with the longitudes and latitudes as already given. By calculation from the longitudes and latitudes as given, the distance was found to be 16° 52′, or 1166½ statute miles. By plotting in equatorial projection, and measuring with protractor No. II, the distance was found to be 16° 53′ (figure 22 shows both New York and New Orleans in equatorial projection). By plotting four different times in meridian projection, using the method shown in figure 23, page 34, the distance as measured by protractor No. II was found to be 16° 53′, 16° 53′, 16° 53′, and 16° 58′. It should be explained that in three cases (a zero point of the protractor being at New York) New Orleans fell just short of coincidence with the 17° line of the protractor; hence the position was recorded as 16° 55′. In one case it was on the 17° line. From these readings 2′ were deducted to allow for the shrinkage of the celluloid protractor. It is in part accidental that four of these determinations, 16° 53′, came so close to the calculated value, 16° 52′; still it is seldom that an error of 6′ is made in plotting and measuring on a meridian projection, and the average of measurements of a similar nature which the writer has made would be not over 4′, probably not over 3′, from the truth. Consider for a moment the discrepancies between measurements as made on ordinary maps and those made by means of the stereographic projection. On maps including the United States alone, and varying in width from 14 to 26 inches, the measurements ranged from 1170 to 1210 miles, a maximum error of 44 miles; while on a stereographic projection of 14 centimeters (5½ inches) diameter, comprising a whole hemisphere, the greatest error was less than seven miles, the average error being but two miles. To appreciate better the discrepancies of the two methods, compare the size of the whole North American continent, as shown on the stereographic map, plate IV, with an ordinary map of the United States. It seems to the writer that this example furnishes a strong argument for discarding the defective methods of map representation now in general use.

As pointed out on page 35 and shown in plate IV, it is an easy matter to shift a stereographic projection so as to bring any desired point to the center. Figure 25 shows the distribution of some of the meridians and parallels when the intersection of the 95th meridian and the 40th parallel is brought to the center of the stereographic projection. The point 95° W. 40° N. is about in the center of the United States. The map is not reduced, but shows the actual size of the United States as it appears at the center of a stereographic projection, based upon a circle of 14^{cm} diameter. The distance from right to left across this map is 28^{mm} ($1\frac{1}{8}$ inches), and the distance from New York

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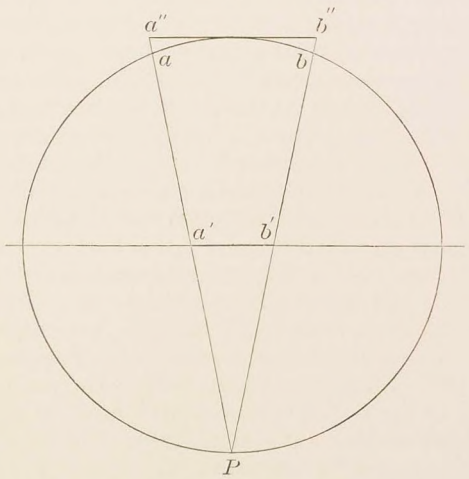

to New Orleans is $10\frac{1}{2}^{mm}$, but little over $\frac{3}{8}$ inch; yet in spite of the very small scale on which the map is made, and also of the fact that the United States is at the center of the hemisphere where it appears the smallest, measurements can be made with the stereographic protractors to within a fraction of a degree. For example, New York to New Orleans was measured by protractor No. II, after making a reduction for shrinkage, as 17° 8′, calculated 16° 52′, a difference of but 16′, or about 18 statute miles.

There is but little distortion, due to the stereographic projection, provided a map does not cover an area larger than that of the United States. Measuring from right to left across the

center of the United States, figure 25, not from east to west along the 40th parallel, but on the arc of a great circle, the edges of the map are about 45° (one-eighth of a circumference) apart. An arc of 45°, *a b*, figure 26, appears in stereographic projection as a line *a′ b′*. A stereographic projection, however, can be made on a tangent plane *a″ b″*, in which case the linear distance from *a″* to *b″* will be twice that of *a′* to *b′*, while the area on a tangent plane will be four times that of a plane passing through the center. As far as distortion is concerned, however, it makes no difference whether a map is made small on a central plane, or with four times the area on a tangent plane, the proportions of the two maps remain the same. Considering the radius of the circle, figure 26, as unity, the distance from *a* to *b*, 45° along the circumference, is 0·785, while from *a″* to *b″* it is 0·828, a difference of only 0·043. These figures indicate that the distortion resulting from the stereographic projection of a small portion of a sphere upon a plane, for example figure 25, can not be very great.

Figure 27 shows a stereographic protractor based upon a 14^{cm} circle, the same as that employed in making the map of the United States, figure 25, and sufficiently large to cover all portions of the map. The only great and small circles which come into consideration are the ones near the center of protractors II and III, plate I and figure 13. The small circles on one half of the protractor indicate degrees, and on the other half statute miles. A semicircular protractor with the small circles indicating either degrees or miles would answer every purpose. Such a protractor would have to be centered on a map of the United States, corresponding to figure 25, at 95° W., 40° N.; that is, at the center of the projection. By turning the protractor, some great circle can be found running through any two points under consideration; and by noting the positions of the points with reference to the small circles, their distance apart may be determined, either in degrees or miles.

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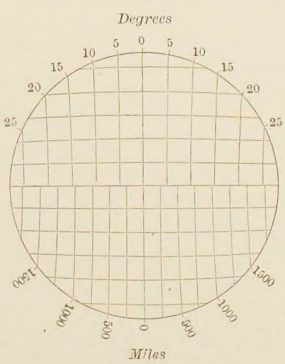

By increasing the width of the map of the United States, figure 25, sixteen times, an ordinary sized atlas sheet, eighteen inches in length, would result. In that case the projection would be based upon a fundamental circle of 224^{cm} (about 7½ feet) diameter, and to plot the stereographically projected meridians and parallels on such a scale would present no difficulty. The

protractor also would have to be increased in like proportion. On such a scale the first degree of the protractor would be $9{\cdot}8^{mm}$ (nearly a centimeter) from the center, and if the small circles of the protractor indicated tenths of a degree (6′), the graduation would be far coarser than that of protractor No. II, plate I. If the small circle graduation of the protractor indicated every fourth minute, the first line would be distant $0{\cdot}652^{mm}$ from the center, while the first-degree line of protractor No. II, plate I, is $0{\cdot}611^{mm}$ from the center. It will be at once evident to any one accustomed to reading scales, that, with divisions $0{\cdot}65^{mm}$ apart, one quarter of the distance between such divisions may be easily estimated, and with care, perhaps, one eighth, which would be equivalent to half a mile on a map of corresponding scale. By making use of a dividing engine, it is possible to construct protractors of any size; and used in connection with carefully plotted maps, geodetic measurements can be made with almost any desired degree of accuracy.

For purposes of navigation of the North Atlantic, for example, it would seem that nothing could be simpler than a stereographic chart with 35° W., 45° N., at the center. On such a chart, 18 inches or more in diameter, all ports and lighthouses could be accurately located. If a navigator can determine his longitude and latitude correctly, which under favorable conditions he is supposed to do within a minute (nautical mile), all that remains to be done to find the distance from any desired point would be to locate his own position on the chart, shift a stereographic protractor so as to bring the two points on the same great circle, and note the position of the two points with reference to the small circles. The great circle, which it would not be necessary to draw upon the chart, would indicate the sailing direction. Compass bearings could be determined by means of an ordinary protractor, as indicated on page 19, figure 12. Calculations involving the use of tables and formulas would not be necessary. An 18-inch chart would probably be quite large enough for practical purposes. At all events, the errors in determining longitude and latitude would probably be as great as those resulting from locating the points on the chart and reading the protractor. The position of a ship from day to day being noted on a stereographic chart, the daily runs could be determined directly by means of a stereographic protractor. Without doubt, vessels have frequently been wrecked because of failure of the commanding officer to make his calculations correctly, but if ships were provided with suitable stereographic charts and protractors such accidents might be avoided. A navigator may make a mistake in locating his position, but, that having been correctly determined and located on a suitable chart, a stereographic protractor

would indicate not only the distance from any point of danger noted on the chart, but also the direction to any desired port. The possibilities of mistakes in calculation, other than those involved in determining the position of the vessel, would not have to be taken into consideration.

Still another method of projection which is in very general use is that of Mercator, invented in 1569.* In this, figure 28, the meridians appear as vertical straight lines equally spaced, and the parallels as horizontal lines so distributed that the

28

Mercator's Projection.

points of the compass preserve the same direction over all the map. In such a projection, areas are very much distorted, especially those near the poles, and distances can not be measured directly. Although the writer does not consider himself competent to pass judgment upon the relative merits of map projections as employed by navigators, he does see many advantages of the stereographic over all other kinds of projection.

It is interesting to note, for example, that the projection of Mercator, designed especially for use in navigation, does not give the navigator the information he most wishes to know, except in a few cases. A straight line drawn on the map from one point to another gives a possible sailing direction between the two points. If the direction is due north or south on any meridian, or east or west on the equator, the course is that of a great circle; hence the shortest possible. In all other cases, however, a course thus plotted as a straight line on the map, and sailed by compass without deviation from the direction indicated, will not correspond to the arc of a great circle; hence will not be the desired shortest possible route. Take as an illustration two ports on the same parallel, for example, Lisbon, 11° 30′ W., 38° 30′ N., and the mouth of Delaware Bay, 75° 0′ W., 38° 30′ N.; how is a mariner to shape his course, provided that no account is taken of ocean currents? To one

* Germain, loc. cit., p. 205.

not accustomed to deal with spherical surfaces and their projections, it would seem, on looking at a map in Mercator's projection, that the course should be due west, and that an east wind would be most favorable. Steaming or sailing thus, however, one would be traveling along the arc of a small circle. In stereographic projection, figure 29, the small circle *S. C.*, crossing all

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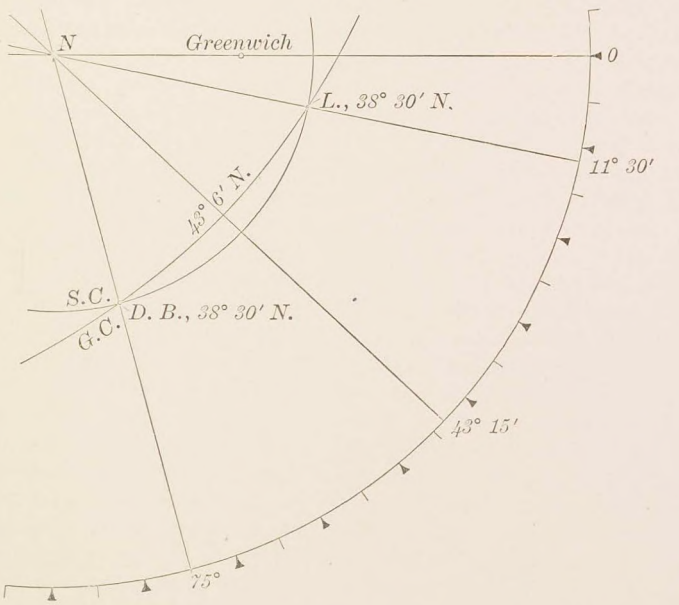
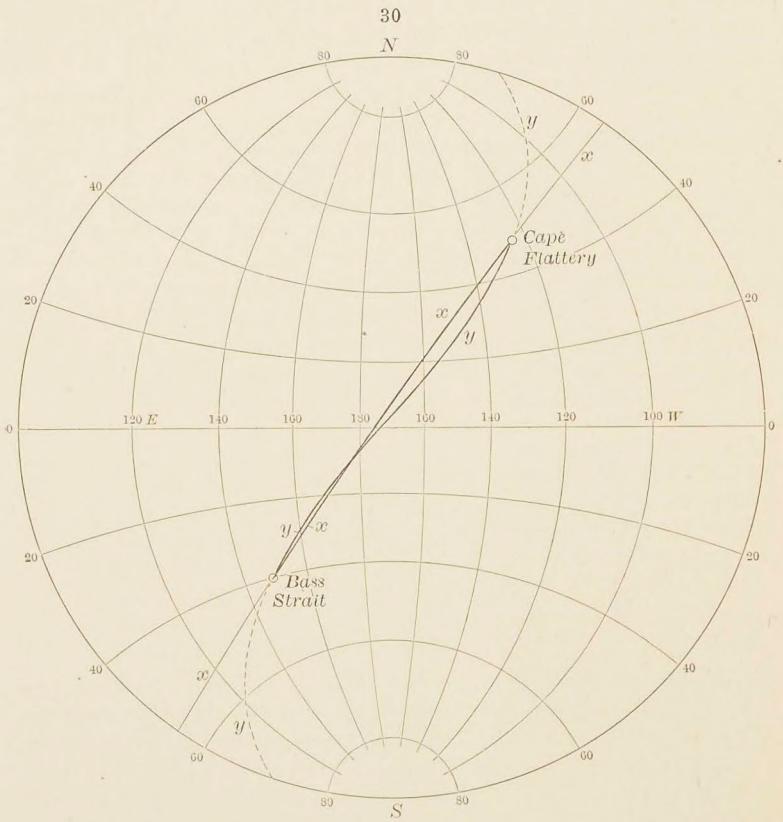


meridians at 90°, is drawn. The shortest course between Lisbon, *L.*, and Delaware Bay, *D.B.*, is, however, the great circle *G.C.*, passing as far north as 43° 6′. Sailing on this great circle, a vessel should be pointed 21° 4′ (plotted 20° 57′) north of west on leaving port, and should gradually change its course so as to cross the intermediate 43° 15′ meridian due west, and then proceed south of west. The distance traveled on the great circle is 48° 38′ (plotted 48° 50′), equal to 2,918 nautical miles, while the distance on the small circle is 2,982 nautical miles, a difference of 64 miles. It is not probable that a steady wind would be an east one swirling around the pole along the arc of a small circle, but, rather, it would be far more likely to travel along the arc of a great circle, crossing the different meridians at slightly different angles. It seems to the writer that these relations, as plotted in stereographic projection, must be far easier to comprehend than if plotted on any other projection.

A line plotted on a sphere so as to intersect all meridians at the same angle, and which appears as a straight line on a Mercator's projection, is known as a rhumb-line, and sailing with the same compass bearings from the place of departure to the place of destination is known as rhumb-sailing. A rhumb-line intersecting the meridians at an



oblique angle (the condition which almost always comes into consideration) gives, when plotted on a sphere, a spiral curve with complex mathematical relations, known as a loxodromic curve. Figure 30 represents the meridians and parallels of a hemisphere in stereographic projection, and two possible sailing routes, from Cape Flattery, 124° 45′ W., 48° 30′ N., the extreme northwestern point of the State of Washington, to the entrance of Bass Strait, 148° E., 40° S., leading to Melbourne, South Australia. The shortest route is the great circle *x x*, which, if followed, necessitates a slight change of compass

bearing, as one meridian after another is passed. The rhumb-course, which intersects all meridians at the same angle, hence maintains the same compass bearing (41° 20′ West of South) throughout, is *y y*. It would be somewhat difficult to estimate the distance traveled in following the rhumb-course, but, judging from the figure, it would not be very much longer than that of the great circle. In regions near the equator the discrepancies between great-circle sailing and rhumb-sailing, both as regards distance and direction, would not be very great. As the polar regions are approached, however, discrepancies become greater. The rhumb-line from Cape Flattery to Bass Strait, figure 30, if continued toward the polar regions, as indicated by the dotted lines *y y*, deviates more and more from the great circle *x x*, and terminates eventually in two spirals, circling around the respective north and south poles, constantly nearing, yet, theoretically, never reaching them.

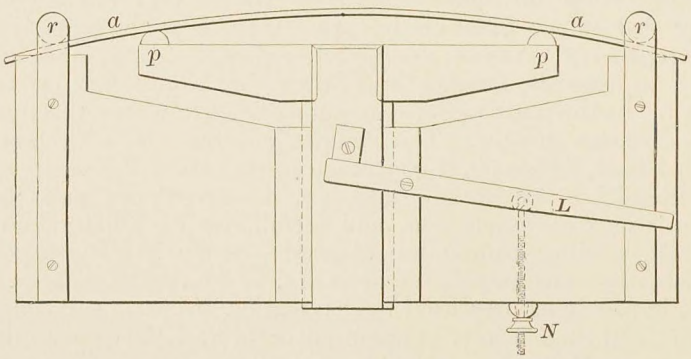
The problems presented by navigation are naturally complicated, and such factors as ocean currents and the probabilities of encountering favorable winds and weather during certain seasons of the year must be taken into consideration. Hence the route along which the *shortest and safest passage may be made* must be selected, rather than the shortest course. The main factor, however, in determining the shortest passage must be the shortest distance from point to point, which may be determined by a stereographic protractor giving great circles, figure 14. The direction of ocean currents and prevailing winds may be represented on a stereographic chart as well as on any other, and it must be a matter of judgment on the part of a navigator to shape his course so as to take advantage, as far as possible, of favorable and avoid unfavorable conditions.

Instruments needed for plotting Stereographic Projections.— For the most part, ordinary drafting instruments may be used. Pencils should be very hard and sharpened to a chisel-like edge, rather than to a point. For locating points very exactly, a fine needle, fastened in a suitable handle, is very useful. Although a puncture made by the needle point may be quite a fraction of a degree in diameter, its center may be regarded as indicating the exact location of any point; and in reading scales and protractors with reference to the puncture, the reading may be easily made from its center. An instrument which is not very generally used and is almost indispensable where great accuracy is required is a beam-compass. Ordinary dividers are not very satisfactory, and when used with an extension bar for constructing large circles the utmost pains must be taken in order to get good results.

For drawing very flat arcs the instrument shown one fourth natural size in figure 31, and designated as the *curved ruler*,

may be employed. The construction of the instrument is very simple. A strip of strong, straight-grained wood, *a a*, is bent by pressure applied in one direction at points *p p*, and in opposite direction at points *r r*. Pressure is applied by means of

31



the lever *L*, which is held in any desired position by means of the nut *N*. The instrument may be quickly adjusted, and the curve between the points *a a*, even when the bending is considerable, corresponds exactly with the arc of a circle. It is no drawback that the wooden strip *a a* becomes permanently bent when left for some time under pressure in the instrument. It is only necessary to withdraw the strip and reverse it, in order to draw a very flat arc, almost a straight line. A curved ruler, in principle like the one figured, but made of metal, was first described by Wulff.* Later Fedorow elaborated the instrument somewhat.† The writer has a metal instrument, made after the Fedorow model by Fuess of Steglitz, near Berlin, but has not found it as satisfactory as the one made of wood. This, however, is only because of certain minor defects. The metal strip in the Fuess instrument, corresponding to *a a* of figure 31, is so high that it is difficult to use a pencil, and almost impossible to use a ruling-pen, in contact with it. Moreover, it is so highly polished that reflections from it are very annoying.

32

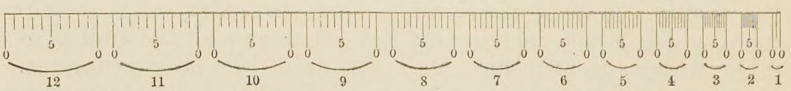
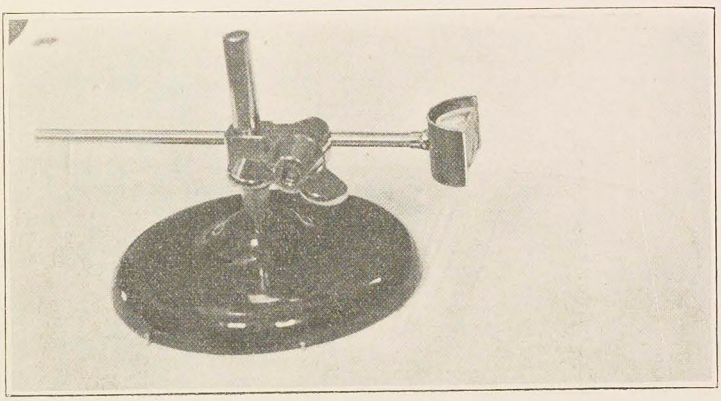


Figure 32, which may be designated as the *scale of decimal parts*, gives a series of spaces (millimeters as indicated by the

* Zeitschr. für Kryst., xxi, p. 253, 1892.
† Zeitschr. für Kryst., xxi, p. 618, 1893.

numbers), each divided into tenths, with the exception of the smallest space. It may be necessary to have spaces of millimeters and a half similarly subdivided, which must be determined by experience. For use, the *scale of decimal parts* will be printed on a card. It is to be used in locating points on charts corresponding to plates II and III, where the meridians and parallels are given for every tenth degree only. For example, to locate Queenstown, 8° 15′ W., 51° 50′ N., on plate II, its longitude is determined by the diam-

33

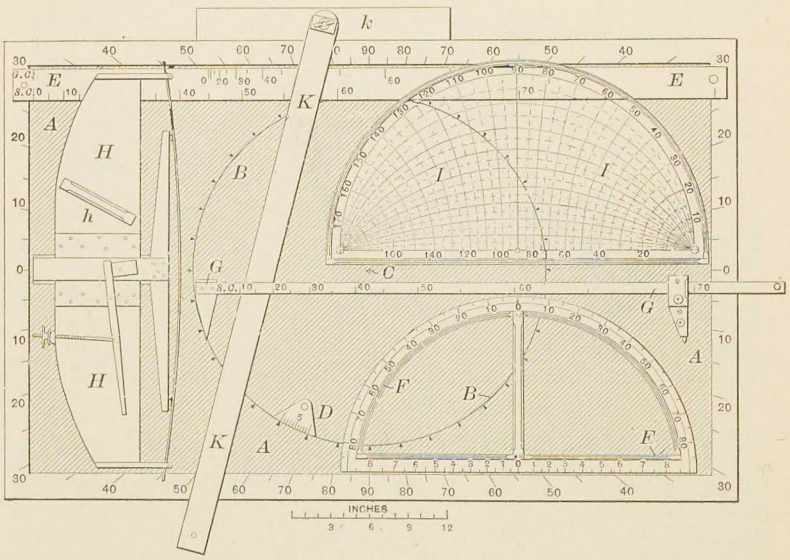


eter 8° 15′ W., and a short line on this diameter, drawn between the parallels 50° and 60° N., locates the place in part. Making use of the *scale of decimal parts*, figure 32, one of the spaces (the 7^{mm} one) will approximate closely to the linear distance between the 50th and 60th parallels, and the location of Queenstown, nearly 2° north of 50°, can be determined with considerable accuracy. Similarly, on a meridian projection, plate III, the approximate location of any desired point between the ten-degree spaces of the meridians and parallels can be quickly ascertained.

Finally, in fixing locations exactly by means of a needle point, with reference either to the scale on the base line of protractor No. I, or to the scale of decimal parts, figure 32, it is convenient to use a lens of low magnifying power, for example, one of two-inch focus. A reading glass or pocket lens may be employed, but, still better, a lens cut in two and mounted on a stand, so that when placed upon the drawing or chart it will be at the proper distance above the paper and does not need to be held. By using only half a lens, the needle-point

holder can be held vertical, and, on a diameter or arc previously drawn, a puncture can be made which is centered very exactly with reference to the graduations of the scales and protractors. A cheap lens is trying to the eyes, as it distorts the lines of the scales, therefore persons making much use of the stereographic projection will find it greatly to their advantage to have a half-lens made from an aplanatic triplet. Figure 33 represents a half-lens made for the writer from a Hastings triplet, two-inch focus, by the Bausch and Lomb Company of Rochester, N. Y. This form of lens will doubtless be convenient for many kinds of work.

34



Black-board Demonstration of the Stereographic Projection.—In explaining the projection to an audience or class, it may be convenient to make use of a black-board, and with a few implements, demonstrations can be made very quickly and accurately.

Figure 34 represents the black-board equipment employed by the writer, in about one fourteenth its natural size. The black-board, *AAA*, is of slate, on which is scratched the circle *BB*, with center *C*. The circle has a radius of 35^{cm}, and its graduation gives every tenth degree. The circle and graduation marks need not be conspicuous, and they interfere in no way with the ordinary uses of the black-board. In case a permanent circle, scratched or painted on a black-board, is objectionable, a center point may be fixed, and degrees on any circle described about it may be taken from graduation marks

painted on the frame of the black-board, as shown. *D* is a scale, drawn on cardboard, which may be used for locating degrees between the ten-degree marks of the graduated circle. *E* is a ruler or straight-edge of light wood, on one side of which are the scales giving the radii of great circles, *G.C.* and vertical small circles, *S.C.*, as shown ; while on the other side are the two scales corresponding to Nos. 3 and 4 of figure 3, page 6. For most operations a smaller ruler, which is not shown, is used. It is provided with the same scales as *E*. A semicircular protractor having the same radius as the circle is shown at *F*, with stereographically projected degree graduations upon its base line. A beam-compass is shown at *G*,* having one fixed arm ending in a metal point, and a sliding arm carrying a crayon. The sliding arm has two screw nuts, one for clamping the arm to the beam, the other for clamping the crayon. The beam may be graduated, the scale on one side giving the radii of vertical small circles, *S.C.*, as shown, while the scale giving the radii of great circles is on the other side. Another compass, exactly like the one shown, except having a shorter arm, is employed in most operations. A large curved ruler is shown at *H*, provided with a handle *h* for holding it in position against the black-board. A stereographic protractor, drawn with India ink on transparent celluloid, is shown at *I*. The face of the protractor is covered with a thin sheet of celluloid to protect the lines, and the two sheets are fastened by screws to a light semicircular frame of wood. *K* is a long, light blade of wood which may be clamped at any angle to the cross-piece *k* by means of a thumb-screw. With this instrument, lines of any desired inclination, taken from the graduation of the circle, may be drawn on any part of the black-board. Although the instrument is not needed for constructing stereographic projections, it has been included with the others in order to make the description of the black-board instruments complete. It is used for solving problems in crystallography by means of graphical methods which will be described in a later communication.

Although the black-board implements are graduated only to every fifth degree, quite accurate work can be done with them. For example, in solving the problems given on pages 25 to 30, the maximum error was 77′, the average error 30′. In making such measurements as from Queenstown to New York or Rio

* A beam-compass of the form shown is an excellent instrument for black-board demonstrations, far superior to the black-board compasses ordinarily employed. If made with scales giving inches on one side of the beam and centimeters on the other, so that circles of any desired radius may be drawn, it would be a very useful instrument, not only in the school- or lecture-room, but in offices of architects and designers. Application has been made for a patent for this instrument.

de Janeiro, pages 33 and 34, results are often within a quarter of a degree of the truth, seldom more than half a degree out.

Conclusion.—Starting with the very simple idea of making a protractor with stereographically projected degrees upon its base line, one possibility after another has presented itself, involving, as stated at the outset, no new mathematical principles, but leading, as the writer believes, to very important results. The main features of this article are the development of the scales shown in figure 3, by means of which it is possible to make stereographic projections very quickly and accurately; and the discovery of the transparent stereographic protractors. The writer can not learn that any one has ever made use of such protractors, and accordingly has made application for patents to control their manufacture and sale. Up to the present time, protractors have been made to conform to a circle of 14^{cm} diameter only, but any demand for protractors and scales based on a larger circle can be easily satisfied. With accurately engraved stereographic plates and protractors, based on a circle of 12 or 18 inches diameter, for example, very accurate plotting and measuring could be done. Many have expressed surprise that on a circle of only 14^{cm} diameter, representing a whole hemisphere, such accurate measurements can be made as those cited in this article. It may be stated, however, that the method in itself is *absolutely exact*, and errors are wholly due to inability to work accurately on so small a scale as the one adopted, and with plates and protractors graduated to degrees only. Based on a circle of 14^{cm} diameter, a stereographically projected degree at the periphery and at the center is represented by distances of about $1{\cdot}2$ and $0{\cdot}6^{mm}$, respectively. Of these distances, one ought to be able to estimate about one tenth of the former and one quarter of the latter; hence, a point carefully located near the periphery ought not to be more than one tenth of a space, or six minutes, out of the way, and the chances are that it will be correct within three minutes, while near the center a point ought to be located at least within a quarter of a degree and probably within ten minutes of the truth. The possibilities as thus stated correspond closely with actual tests of the method. If the plotting is of a simple nature and made near the periphery, and especially if the measurement can be made with one reading of the protractor, the result is rarely more than $6'$ and averages less than $4'$ from the truth. If, on the other hand, the plotting is done near the center of the circle, and measurements necessitate two readings of the protractor, errors amounting to a quarter of a degree must be expected, but the average will be less than $10'$ from the truth.

The stereographic projection is admirably adapted for representing all kinds of relations pertaining to a sphere. It is very difficult, almost impossible, to plot and measure accurately on a spherical surface, and a sphere which may be used for such purposes is rarely at hand; hence, to be able to plot and measure accurately on a flat surface is a great advantage. The writer hopes, therefore, that as a result of this article the stereographic projection will become more widely known and appreciated, and that it will prove more generally useful. Crystallographers have employed the stereographic projection not only for indicating the distribution of crystal faces, but especially for showing zonal (great circle) relations and the spherical triangles which when solved determine the interfacial angles. The stereographic protractors will now give to the projection a new and far more important significance. Crystal forms being plotted according to some fixed scale, desired angles may be measured with sufficient accuracy for most purposes by the protractors. Protractor No. IV, page 22, will indicate zonal relations; and the solution of many other problems follow, which will form the basis of a later communication.

The application of the stereographic projection to astronomical problems are very numerous, and doubtless some astronomers will find their work simplified by using scales and protractors similar to the ones described in this article.

It would seem as though no course in spherical trigonometry could be quite complete without reference to the possibilities which the stereographic projection offers for the solution of all the problems presented.

Before closing this communication the writer takes pleasure in calling attention to a recent article by Professor E. von Fedorow of Petrowsko-Rasumowskoje, near Moscow, on a "*Universalgoniometer mit mehr als zwei Drehaxen und genaue graphische Rechnung.*"* In this communication, Professor Fedorow describes a modification of the two-circle goniometer, by means of which spherical triangles may be solved accurately by purely mechanical methods. This is accomplished by having two reflecting surfaces which may be set at any angle (*künstlicher Krystall*), and which take the place of a crystal on the instrument. By obtaining appropriate reflections from the surfaces, which necessitates the turning of certain circles, the problems are solved, the angles being read from verniers accompanying the graduations of the circles. Although it is shown that the instrument is capable of giving exact results, the applications of the Fedorow method of solv-

* Zeitschr. für Kryst., xxxii, p. 464, 1890.

ing triangles will probably be limited, since an expensive instrument must be procured, which not only must be kept in perfect adjustment, but can be used only by persons who thoroughly understand the workings of its several parts. Then, too, the instrument does not give a map or chart, which is such an important feature of the stereographic projection.

Although it is not customary for the writer of a scientific article to advertise, there are many who may wish to apply stereographic methods to the solution of problems in which they are especially interested, and therefore will find it convenient to make use of the printed sheets, scales, and protractors described herewith. Accordingly arrangements have been made to keep a supply of the necessary articles in stock, and, upon application either to the writer or to E. L. Washburn & Co. of New Haven, a price list will be sent.

If the method fulfills the writer's expectations, there will be in time a demand for scales and protractors of larger size and finer graduation, based, for example, on circles of 12 or 18 inches diameter. Moreover, with the experience already gained, there ought to be no serious difficulty in making them very accurately.

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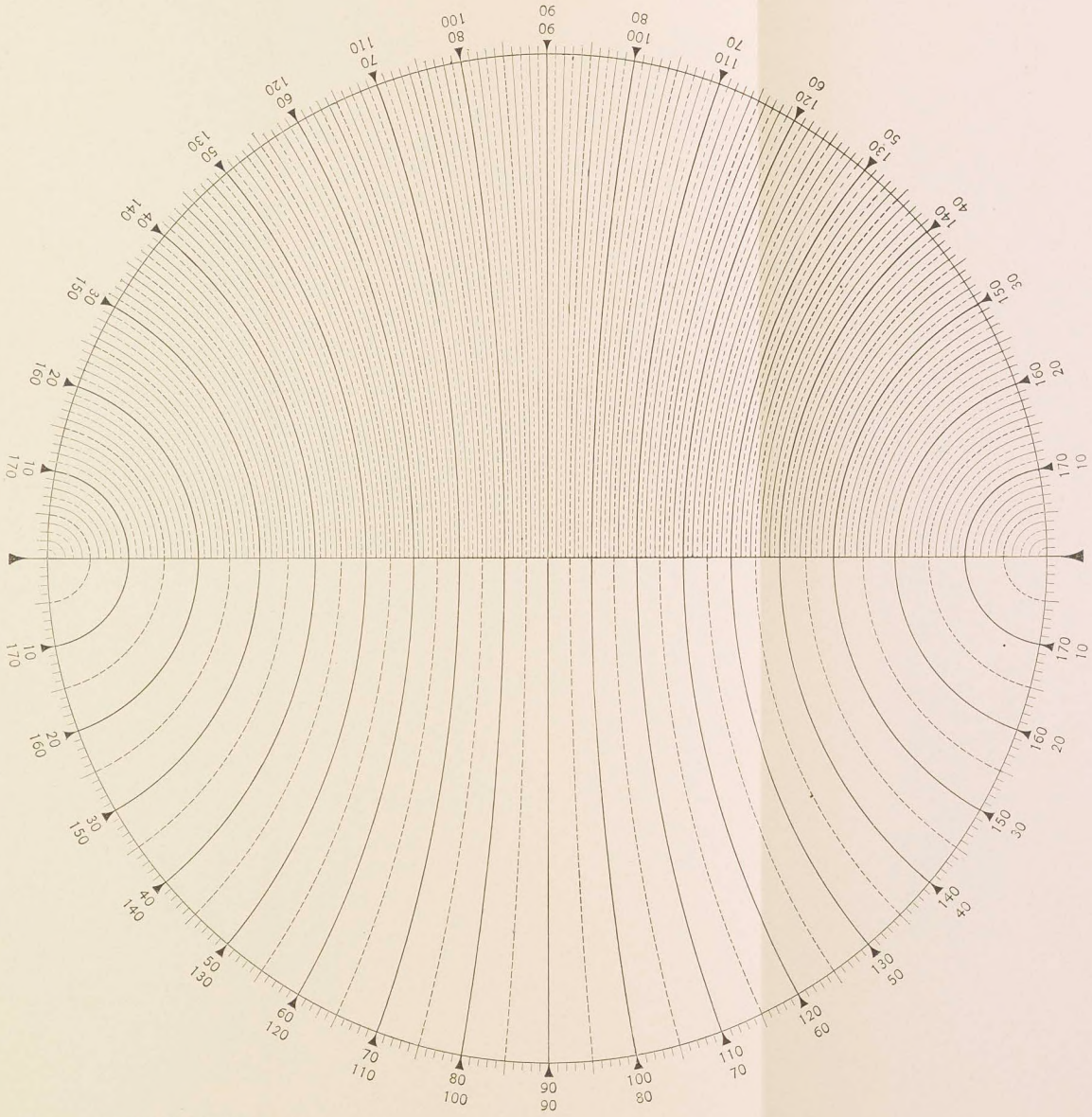
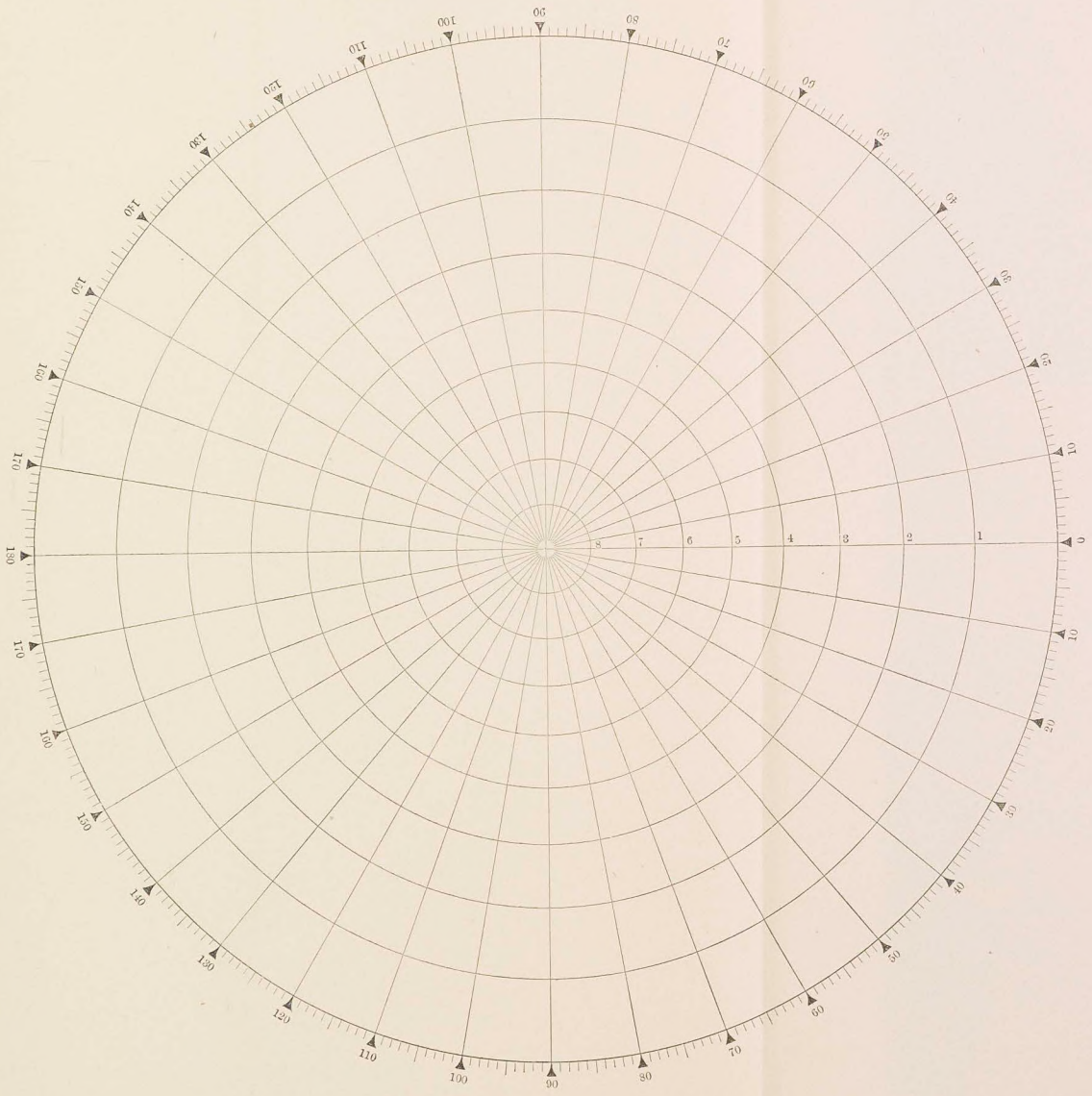


Patented Feb. 5, 1901.

STEREOGRAPHIC PROTRACTOR No. II, FOR MEASURING THE ARCS OF GREAT CIRCLES.

Printed from the original engine-divided plate.



MERIDIANS AND PARALLELS IN STEREOGRAPHIC PROJECTION. EQUATORIAL PROJECTION.

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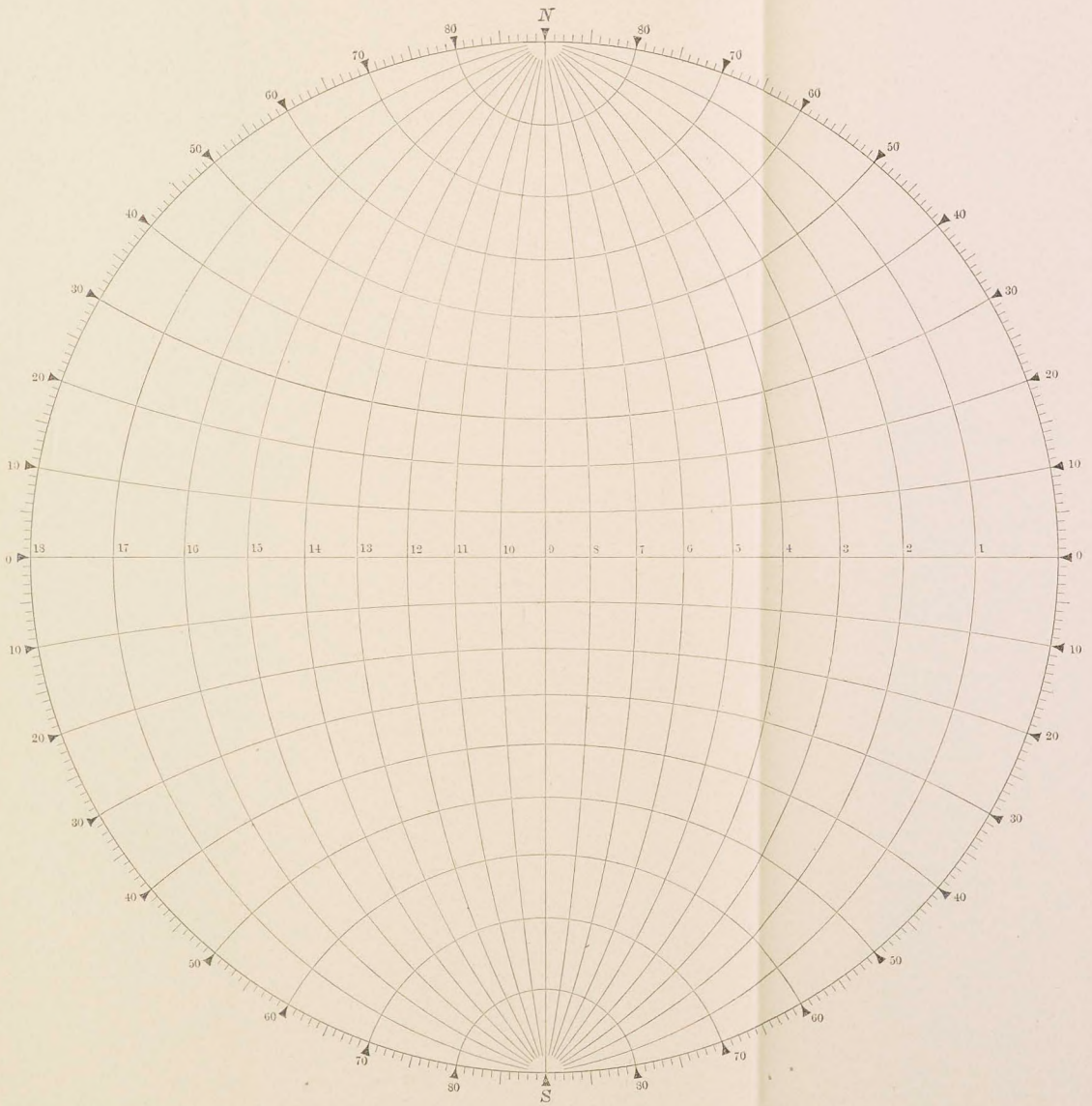
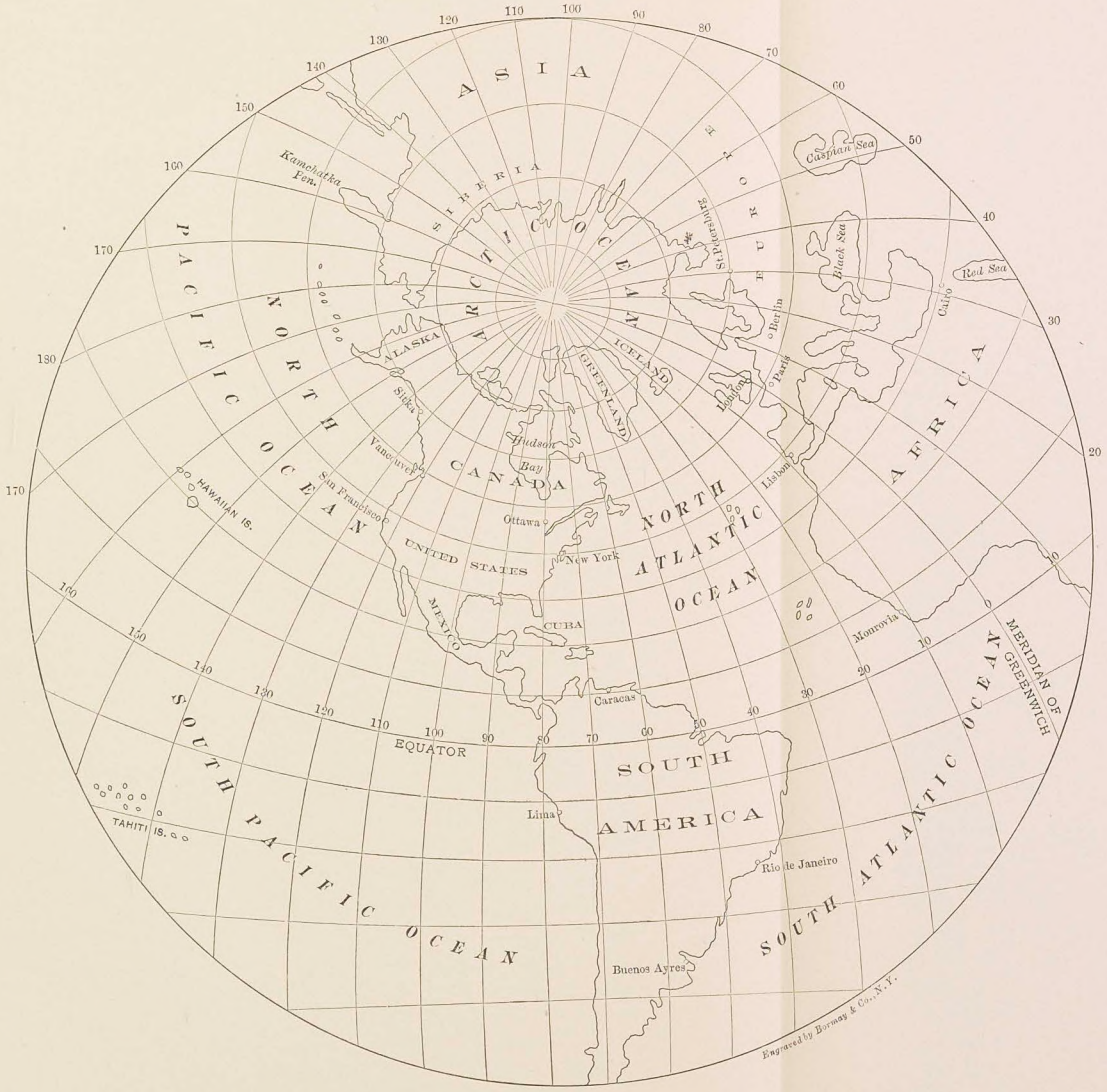


MERIDIANS AND PARALLELS IN STEREOGRAPHIC PROJECTION. MERIDIAN PROJECTION.

Printed from the original engine-divided plate.

Plate IV.



HEMISPHERE IN STEREOGRAPHIC PROJECTION,

With 75° W. 40° N., at the center.